

On the sizes of burnt and fireproof components for fires on a large Cayley tree

Cyril Marzouk

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland. E-mail: cyril.marzouk@math.uzh.ch

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Abstract. We continue the study initiated by Jean Bertoin in 2012 of a random dynamics on the edges of a uniform Cayley tree with n vertices in which, successively, each edge is either set on fire with some fixed probability p_n or fireproof with probability $1 - p_n$. An edge which is set on fire burns and sets on fire its flammable neighbors, the fire then propagates in the tree, only stopped by fireproof edges. We study the distribution of the proportion of burnt and fireproof vertices and the sizes of the burnt or fireproof connected components as $n \rightarrow \infty$ regarding the asymptotic behavior of p_n .

Résumé. On poursuit l'étude initiée par Jean Bertoin en 2012 d'une dynamique aléatoire sur un arbre de Cayley uniforme avec n sommets dans laquelle, successivement, chaque arête est soit enflammée avec une probabilité p_n fixée, soit ignifugée avec probabilité $1 - p_n$. Une arête enflammée brûle et enflamme ses arêtes voisines inflammables, le feu se propage alors dans l'arbre, stoppé uniquement par les arêtes ignifugées. On étudie la loi de la proportion de sites ignifugés et brûlés ainsi que la taille des composantes connexes ignifugées et brûlées lorsque $n \rightarrow \infty$ selon le comportement asymptotic de p_n .

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1. Introduction and main results

We recall the definition of the fire dynamics introduced by Bertoin [5]. Given a tree of size n and a number $p_n \in [0, 1]$, we consider the following random dynamics: initially every edge is flammable, then successively, in a random uniform order, each edge is either fireproof with probability $1 - p_n$ or set on fire with probability p_n . In the latter case, the edge burns, sets on fire its flammable neighbors and the fire propagates instantly in the tree, only stopped by fireproof edges. An edge which has been burnt because of the propagation of fire is not subject to the dynamics thereafter. The dynamics continue until all edges are either burnt or fireproof. A vertex is called fireproof if all its adjacent edges are fireproof and called burnt otherwise; we discard fireproof edges that have at least one burnt extremity and thus get two forests: one consists of fireproof trees and the other of burnt trees. See Figure 1 for an illustration. We study the asymptotic behavior of the size of these two forests and of their connected components as the total size of the tree tends to infinity.

In this work, we assume that the tree is a uniform Cayley tree of size n , denoted by \mathbf{t}_n , i.e. picked uniformly at random amongst the n^{n-2} different trees on a set of n labeled vertices, say, $[n] = \{1, \dots, n\}$. For this model, the system exhibits a phase transition as it is shown by Bertoin [5]. Theorem 1 in [5] is stated in the case where $p_n \sim cn^{-\alpha}$ with $c, \alpha > 0$ but extends verbatim as follows: denote by I_n and B_n respectively the total number of fireproof and burnt vertices of \mathbf{t}_n , then

- (i) If $\lim_{n \rightarrow \infty} n^{1/2} p_n = \infty$ (subcritical regime), then $\lim_{n \rightarrow \infty} n^{-1} I_n = 0$ in probability.

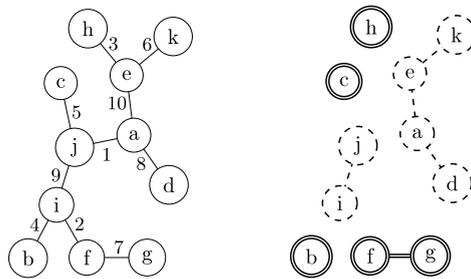


Fig. 1. Given a tree and a uniform enumeration of its edges on the left, if the edges set on fire are the 6th and the 9th, we get the two forests on the right where dashed lines stand for “burnt” and double lines for “fireproof.”

- (ii) If $\lim_{n \rightarrow \infty} n^{1/2} p_n = 0$ (supercritical regime), then $\lim_{n \rightarrow \infty} n^{-1} B_n = 0$ in probability.
- (iii) If $\lim_{n \rightarrow \infty} n^{1/2} p_n = c$ for some $c > 0$ (critical regime), then $\lim_{n \rightarrow \infty} n^{-1} I_n = D(c)$ in distribution where

$$P(D(c) \in dx) = \frac{c}{\sqrt{2\pi x(1-x)^3}} \exp\left(-\frac{c^2 x}{2(1-x)}\right) dx, \quad 0 < x < 1. \tag{1}$$

The aim of this paper is to improve these three convergences. For the first two regimes, we prove a convergence in distribution to a non-trivial limit under an appropriate scaling of I_n and B_n respectively, see the statements below. For the critical regime, we prove the joint convergence in distribution of the number of fireproof vertices and the sizes of the burnt subtrees, ranked in non-increasing order; the precise statement requires some notations and is postponed to Section 3, see Theorem 3 there. We next state our main result concerning the subcritical regime.

Theorem 1. *Suppose that $\lim_{n \rightarrow \infty} n^{1/2} p_n = \infty$. Then*

$$\lim_{n \rightarrow \infty} p_n^2 I_n = Z^2 \quad \text{in distribution,}$$

where Z is a standard Gaussian random variable.

Consider then the supercritical regime; as we are interested in the asymptotic behavior of B_n we assume that $p_n \gg n^{-1}$ so that the probability that no fire occurs is $(1 - p_n)^{n-1} \rightarrow 0$.

Theorem 2. *Suppose that $\lim_{n \rightarrow \infty} n^{1/2} p_n = 0$ and $\lim_{n \rightarrow \infty} n p_n = \infty$. Then*

$$\lim_{n \rightarrow \infty} (n p_n)^{-2} B_n = Z^{-2} \quad \text{in distribution,}$$

where Z is a standard Gaussian random variable.

Remark 1. *Let Z be a standard Gaussian random variable. One can check from (1) that*

$$\lim_{c \rightarrow \infty} c^2 D(c) = Z^2 \quad \text{and} \quad \lim_{c \rightarrow 0} c^{-2} (1 - D(c)) = Z^{-2} \tag{2}$$

in distribution. Very informally, if we write $I_n(p_n)$ for the number of fireproof vertices of \mathbf{t}_n when the probability to set on fire a given edge is p_n , and similarly $B_n(p_n)$, then (iii) above shows that for every $c \in (0, \infty)$ fixed,

$$I_n(c n^{-1/2}) \approx n D(c) \quad \text{and} \quad B_n(c n^{-1/2}) \approx n (1 - D(c)).$$

From (2), one is tempted to write more generally for $p_n \gg n^{-1/2}$,

$$I_n(p_n) \approx n D(n^{1/2} p_n) \approx p_n^{-2} Z^2,$$

and for $p_n \ll n^{-1/2}$,

$$B_n(p_n) \approx n(1 - D(n^{1/2} p_n)) \approx (np_n)^2 Z^{-2}.$$

However, it does not seem clear to the author how to prove respectively Theorems 1 and 2 from this sketch. Indeed the argument in [5] does not enable one to deal with the sub or supercritical regime and the proofs given here are different from that of (i), (ii) and (iii).

The rest of this paper is organized as follows. Relying on Pitman [11] and Chaumont and Uribe Bravo [7], we briefly discuss in Section 2 the existence of a conditional distribution for the sequence of the ranked sizes of the jumps made during the time interval $[0, 1]$ by a certain subordinator, say, σ , conditionally given the value of the latter at time 1. We also prove the continuity of this conditional distribution in the terminal value $\sigma(1)$, which will be used to derive our first result.

We then focus on the critical regime in Section 3. Motivated by the proof of (iii) in [5], we associate with the Cayley tree \mathbf{t}_n its cut-tree and view the fire dynamics as a point process on the latter. Using the ideas of Aldous and Pitman [3], we show that the marked cut-tree converges to the Brownian continuum random tree (CRT) endowed with a slight modified version of the point process obtained in [3]. This yields the joint convergence of the number of fireproof vertices and the sizes of the burnt connected components to the masses of the components of the CRT logged at the atoms of the point process. Using a second approximation of the CRT with finite trees, we further express this limit as a mixture of the jumps of the previous subordinator σ conditioned on the value of the latter at time 1, with a mixing law $D(c)$.

We prove Theorem 1 in Section 4. For this, we shall see that, with high probability, the remaining forest after the first fire has a total size of order p_n^{-2} and so have its largest trees. Note that the dynamics then continue on each subtree independently. Informally, the smallest ones do not contribute much and may be neglected, while the dynamics on the largest subtrees are now critical. A slight generalization of (iii) then yields an asymptotic for the number of fireproof vertices in each subtree and so for the total number of fireproof vertices.

Finally, we prove Theorem 2 in Section 5. Consider the sequence of the sizes of the burnt subtrees, ranked in order of appearance, and all rescaled by a factor $(np_n)^{-2}$. We prove that the latter converges in distribution for the ℓ^1 topology, from which Theorem 2 follows readily. To this end, we first show for every integer j the joint convergence for the size of the j first burnt subtrees; then we show that, taking j large enough, the next trees are arbitrary small.

2. Preliminaries on subordinators and bridges

Let $(\sigma(t), t \geq 0)$ be the first-passage time process of a linear Brownian motion: σ is a stable subordinator of index $1/2$ such that

$$E[\exp(-q\sigma(t))] = \exp(-t\sqrt{2q}) \quad \text{for any } t, q \geq 0. \tag{3}$$

Let $J_1 \geq J_2 \geq \dots \geq 0$ be the ranked sizes of its jumps made during the time interval $[0, 1]$. We need to make sense of the conditional distribution of the sequence $(J_i)_{i \geq 1}$ conditionally given the null event $\{\sigma(1) = z\}$ in the set $\ell^1(\mathbf{R})$ of real-valued summable sequences.

From the Lévy–Itô decomposition, we know that σ is a right-continuous, non-decreasing process which increases only by jumps – we say that σ is a pure jump process – and that the pairs (t, x) induced by the times and sizes of the jumps are distributed as the atoms of a Poisson random measure on $[0, 1] \times (0, \infty)$ with intensity $(2\pi x^3)^{-1/2} dt dx$. Denote by $(P_i)_{i \geq 1}$ a size-biased permutation of the sequence $(J_i / \sum_k J_k)_{i \geq 1}$. Pitman [11] gives an inductive construction of a regular conditional distribution for $(P_i)_{i \geq 1}$ given $\{\sum_k J_k = z\}$ for arbitrary $z > 0$. The latter determines the conditional distribution of $(J_i / \sum_k J_k)_{i \geq 1}$ given $\{\sum_k J_k = z\}$ called Poisson–Kingman distribution. Descriptions of finite-dimensional distributions can be found in Perman [9] or in Pitman and Yor [12]. Our purpose here is to check that these distributions depend continuously on the variable z .

Proposition 1. *The conditional distribution of the ranked jump-sizes $(J_i)_{i \geq 1}$ given $\{\sigma(1) = z\}$ is continuous in z .*

Proof. In the recent work of Chaumont and Uribe Bravo [7], sufficient conditions on the distribution of a Markov process $(X_t, t \geq 0)$ in a quite general metric space are given in order to make sense of a conditioned version of $(X_s, 0 \leq s \leq t)$ given $\{X_0 = x \text{ and } X_t = y\}$. The latter is called Markovian bridge from x to y of length t and its law is denoted by $\mathbf{P}_{x,y}^t$. The process σ fulfills the framework of their Theorem 1 and Corollary 1, it follows that the bridge laws $\mathbf{P}_{0,z}^1$ are well defined and continuous in z for the Skorohod topology. Thanks to Skorohod's representation theorem, the claim thus reduces to the deterministic result below. \square

Let f, f_1, f_2, \dots be functions defined from $[0, 1]$ to $[0, \infty)$ which are non-decreasing, right-continuous and null at 0. Denote by $j_1 \geq j_2 \geq \dots \geq 0$ the ranked sizes of the jumps of f and respectively, $j_1^{(n)} \geq j_2^{(n)} \geq \dots \geq 0$ that of f_n for every $n \geq 1$.

Lemma 1. *Suppose that f_n converges to f for the Skorohod topology. Then*

- (i) *For any integer N , $(j_1^{(n)}, \dots, j_N^{(n)})$ converges to (j_1, \dots, j_N) in \mathbf{R}^N .*
- (ii) *If f is a pure jump function, then $(j_k^{(n)})_{k \geq 1}$ converges to $(j_k)_{k \geq 1}$.*

Proof. For the first claim, suppose first that f has infinitely many jumps. We may, and do, assume that N is such that $j_N > j_{N+1}$. For any t , denote by $\Delta f(t) := f(t) - f(t^-)$ the size of the jump made by f at time t and similarly Δf_n for every $n \geq 1$. Upon changing the time scale using a sequence of increasing homeomorphisms from $[0, 1]$ onto itself which converges uniformly to the identity, we may assume that f_n converges to f uniformly. This does not affect the jump-sizes of f_n . Then $\Delta f_n(t)$ converges to $\Delta f(t)$ for every t and (j_1, \dots, j_N) are limits of N jumps of f_n . Moreover, these jumps are $(j_1^{(n)}, \dots, j_N^{(n)})$ for n large enough since, for any $\varepsilon \in (0, j_N - j_{N+1})$, for any n large enough, as f_n converges to f uniformly, it admits no other jump larger than $j_{N+1} + \varepsilon/2 < j_N - \varepsilon/2$. If f has only finitely many jumps, say, N , this reasoning yields the convergences $(j_1^{(n)}, \dots, j_N^{(n)}) \rightarrow (j_1, \dots, j_N)$ and $j_k^{(n)} \rightarrow 0$ for any $k \geq N + 1$.

For the second claim, we write for any integer N fixed,

$$\sum_{k=1}^{\infty} |j_k^{(n)} - j_k| \leq \sum_{k=1}^N |j_k^{(n)} - j_k| + \sum_{k=N+1}^{\infty} j_k + \sum_{k=N+1}^{\infty} j_k^{(n)}.$$

As $n \rightarrow \infty$, the first term tends to 0 from (i). Let $\varepsilon > 0$ and fix N such that $\sum_{k=N+1}^{\infty} j_k < \varepsilon$. Since f is a pure jump function, we have $f(1) = \sum_{k=1}^{\infty} j_k$ and so $\sum_{k=1}^N j_k \geq f(1) - \varepsilon$. Finally, since $\lim_{n \rightarrow \infty} f_n(1) = f(1)$, we conclude that $\sum_{k=N+1}^{\infty} j_k^{(n)} \leq f_n(1) - \sum_{k=1}^N j_k^{(n)} \leq 2\varepsilon$ for n large enough. \square

3. Asymptotic size of the burnt subtrees in the critical regime

Fix $c \in (0, \infty)$ and consider the critical regime $p_n \sim cn^{-1/2}$ of the fire dynamics on \mathbf{t}_n . Let κ_n be the number of burnt subtrees, $b_{n,1}, \dots, b_{n,\kappa_n}$ their respective size, listed in order of appearance, and finally $b_{n,1}^* \geq \dots \geq b_{n,\kappa_n}^*$ a non-increasing rearrangement of the latter. We can now state the main result of this section.

Theorem 3. *For all continuous and bounded maps $f : (0, 1) \rightarrow \mathbf{R}$ and $F : \ell^1(\mathbf{R}) \rightarrow \mathbf{R}$, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \left[f \left(\frac{I_n}{n} \right) F \left(\frac{b_{n,1}^*}{n}, \dots, \frac{b_{n,\kappa_n}^*}{n} \right) \right] \\ &= \int_0^1 f(x) \mathbb{E} \left[F \left(\frac{(1-x)J_1}{\sigma(1)}, \frac{(1-x)J_2}{\sigma(1)}, \dots \right) \middle| \sigma(1) = \frac{1-x}{c^2 x^2} \right] \mathbb{P}(D(c) \in dx), \end{aligned}$$

where σ is a subordinator distributed as (3) and $\mathbb{P}(D(c) \in dx)$ is defined in (1).

Note that, taking $F \equiv 1$, this recovers the result (iii) in the [Introduction](#); moreover, since $\sum_i J_i = \sigma(1)$, it strengthens (iii) by giving the decomposition of the burnt forest conditionally given its total size.

The proof is divided in two parts. As discussed in the [Introduction](#), we view the fire dynamics on \mathbf{t}_n as a mark process on the associated cut-tree $\text{Cut}(\mathbf{t}_n)$, which translates the vector $n^{-1}(I_n, \mathbf{b}_{n,1}^*, \dots, \mathbf{b}_{n,k_n}^*)$ into the proportion of leaves of the trees in the forest obtained by logging $\text{Cut}(\mathbf{t}_n)$ at the marks. We prove that the marked tree $\text{Cut}(\mathbf{t}_n)$, properly rescaled, converges to the CRT endowed with a certain point process; it follows that the previous vector converges to the masses of the trees in the forest obtained by logging the CRT at the atoms of the point process. We then study the distribution of the latter. As direct computations with the CRT seem rather complicated, we approximate the marked CRT by a Galton–Watson tree with Poisson(1) offspring distribution conditioned to have n vertices and endowed with the same mark process as the cut-tree $\text{Cut}(\mathbf{t}_n)$. We refer to Aldous [1] and Aldous and Pitman [3] for prerequisites about the CRT, its logging by a Poisson point process and convergence of conditioned Galton–Watson trees.

3.1. Binary cut-tree, fire dynamics and mark process

Given a tree T_n on a set of n labeled vertices, say $[n] = \{1, \dots, n\}$, we build inductively its cut-tree $\text{Cut}(T_n)$, which is a random rooted binary tree with n leaves. Each vertex of $\text{Cut}(T_n)$ corresponds to a subset (or *block*) of $[n]$, the root of $\text{Cut}(T_n)$ is the whole set $[n]$ and its leaves are the singletons $\{1, \dots, n\}$. We remove successively the edges of T_n in a random uniform order; at each step, a subtree of T_n with set of vertices, say, B , falls into two subtrees with set of vertices, say, B' and B'' respectively; in $\text{Cut}(T_n)$, B' and B'' are the two offsprings of B . Notice that, by construction, the set of leaves of the subtree of $\text{Cut}(T_n)$ generated by some block coincides with this block. See Figure 2 for an illustration.

Let \mathbf{t}_n be a Cayley tree with n vertices and let $\text{Cut}(\mathbf{t}_n)$ be the associated cut-tree. We can encode the fire dynamics on \mathbf{t}_n as a mark process on the vertices of $\text{Cut}(\mathbf{t}_n)$. It is convenient to see fireproof edges of \mathbf{t}_n as deleted, then when an edge is set on fire, the whole subtree that contains it burns instantly and we mark the corresponding block of $\text{Cut}(\mathbf{t}_n)$. The leaves of $\text{Cut}(\mathbf{t}_n)$ cannot be marked as they correspond to singletons in \mathbf{t}_n . Note that if a block of $\text{Cut}(\mathbf{t}_n)$ is marked, its descendants are never marked because the edges of the corresponding subtree of \mathbf{t}_n are no longer subject to the dynamics. The marked blocks of $\text{Cut}(\mathbf{t}_n)$ are exactly the burnt components of \mathbf{t}_n and, as we noticed, their size is the number of leaves of the subtree of $\text{Cut}(\mathbf{t}_n)$ they generate; see Figure 3 for an illustration.

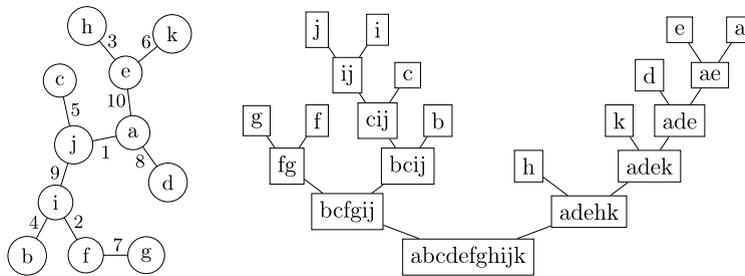


Fig. 2. A tree with the order of cuts on the left and the corresponding cut-tree on the right.

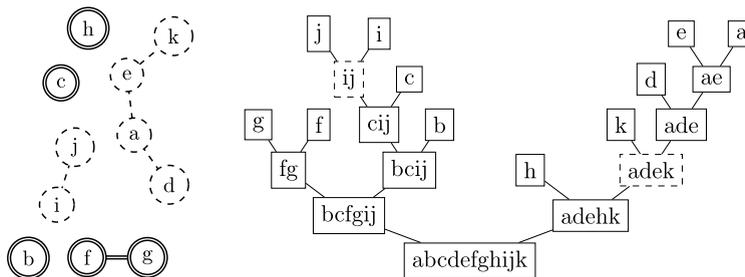


Fig. 3. The forest after the dynamics and the corresponding marked cut-tree.

Given the cut-tree $\text{Cut}(\mathbf{t}_n)$, the mark process can be constructed as follows: at each generation, each internal (i.e. non-singleton) block is marked independently of the others with probability p_n provided that none of its ancestor has been marked, and not marked otherwise. This is equivalent to the following two-steps procedure: mark first every internal block independently with probability p_n , then along each branch, keep only the closest mark to the root and erase the other marks. We will refer to this procedure as the *marking–erasing process* associated with the point process which marks each internal block independently with probability p_n .

3.2. Convergence of marked trees

It will be more convenient to mark $\text{Cut}(\mathbf{t}_n)$ on its edges rather than on its vertices so we shift the marks defined above from a vertex to the edge that connects it to its parent: denote by φ'_n the mark process which marks each edge of $\text{Cut}(\mathbf{t}_n)$ which is not adjacent to a leaf on its mid-point independently with probability p_n , by φ_n the associated marking–erasing process and by $\#(\text{Cut}(\mathbf{t}_n), \varphi_n)$ the vector whose entries count the number of leaves of each tree in the forest obtained by logging $\text{Cut}(\mathbf{t}_n)$ at the marks of φ_n , the root-component first, and the next in non-increasing order.

Let \mathcal{T} be a rooted Brownian CRT, μ its uniform probability “mass” measure on leaves and the usual distance d . The distance induces a “length” measure ℓ , which is the unique σ -finite measure assigning measure $d(x, y)$ to the geodesic path between x and y in \mathcal{T} . Denote by Φ' a Poisson point process with intensity $c\ell(\cdot)$ on the skeleton of \mathcal{T} , Φ the associated marking–erasing process and $\#(\mathcal{T}, \Phi)$ the vector whose entries count the mass of each tree in the forest obtained by logging \mathcal{T} at the atoms of Φ , again the root-component first, and the next in non-increasing order.

Lemma 2. *The vector $n^{-1}\#(\text{Cut}(\mathbf{t}_n), \varphi_n)$ converges in distribution to $\#(\mathcal{T}, \Phi)$ for the ℓ^1 topology.*

We endow the tree $\text{Cut}(\mathbf{t}_n)$ with the uniform distribution on leaves μ_n and the metric d_n given by the graph distance rescaled by a factor $n^{-1/2}$. For every integer $k \geq 1$, denote by $\mathcal{R}_n(k)$ the smallest connected subset of $\text{Cut}(\mathbf{t}_n)$ which contains the root $[n]$ and k i.i.d. leaves chosen according to μ_n ; we call $\mathcal{R}_n(k)$ the tree $\text{Cut}(\mathbf{t}_n)$ reduced to those leaves. Denote similarly by $\mathcal{R}(k)$, the CRT reduced to k i.i.d. elements picked according to μ . We see the reduced trees as finite rooted metric spaces; the proof of Lemma 1 in Bertoin [5] shows that for every $k \geq 1$ fixed,

$$\lim_{n \rightarrow \infty} \mathcal{R}_n(k) = \mathcal{R}(k) \quad \text{in distribution} \tag{4}$$

in the sense of Gromov–Hausdorff. This is equivalent to the convergence of the rooted metric measure spaces $(\text{Cut}(\mathbf{t}_n), d_n, \mu_n)$ to (\mathcal{T}, d, μ) in distribution for the so-called Gromov–Prokhorov topology.

Proof of Lemma 2. Since the scaling factor of $\text{Cut}(\mathbf{t}_n)$ corresponds to p_n , we may, and do, extend (4) to the joint convergence of $\mathcal{R}_n(k)$ and the trace of φ'_n on its edges to $\mathcal{R}(k)$ endowed with a Poisson point process with rate c per unit length on its edges. The same convergence holds when considering the marking–erasing processes; subsequently, for every $k \geq 1$,

$$\lim_{n \rightarrow \infty} \# \mathcal{R}_n(k, \varphi_n) = \# \mathcal{R}(k, \Phi)$$

in distribution, where $\# \mathcal{R}_n(k, \varphi_n)$ denotes the vector whose entries count the number of leaves of each tree in the forest obtained by logging $\mathcal{R}_n(k)$ at the marks induced by φ_n , the root-component first, and the next in non-increasing order, and similarly for $\# \mathcal{R}(k, \Phi)$. Since

$$\lim_{k \rightarrow \infty} k^{-1} \# \mathcal{R}(k, \Phi) = \#(\mathcal{T}, \Phi),$$

it follows from a diagonal argument that for $k_n \rightarrow \infty$ sufficiently slowly as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} k_n^{-1} \# \mathcal{R}_n(k_n, \varphi_n) = \#(\mathcal{T}, \Phi)$$

in distribution. Adapting Lemma 11 of Aldous and Pitman [3] to uniform sampling of leaves instead of vertices, this finally yields

$$\lim_{n \rightarrow \infty} n^{-1} \#(\text{Cut}(\mathbf{t}_n), \varphi_n) = \#(\mathcal{T}, \Phi)$$

in distribution. □

A consequence of this lemma is the following result, which is the first step in the proof of Theorem 3. Recall that I_n stands for the total number of fireproof vertices of the Cayley tree \mathbf{t}_n , κ_n for the number of burnt components and $b_{n,1}^* \geq \dots \geq b_{n,\kappa_n}^*$ for their respective size, ranked in non-increasing order.

Proposition 2. *We have*

$$\lim_{n \rightarrow \infty} n^{-1} (I_n, b_{n,1}^*, \dots, b_{n,\kappa_n}^*) = \#(\mathcal{T}, \Phi) \quad \text{in distribution.}$$

Proof. Recall that the connected components of $\text{Cut}(\mathbf{t}_n)$ that do not contain the root correspond to the burnt subtrees of \mathbf{t}_n , whereas the root-component corresponds to the fireproof forest. Recall also that the number of leaves in $\text{Cut}(\mathbf{t}_n)$ of each component is the number of vertices of the corresponding in \mathbf{t}_n . We then get the identity

$$(I_n, b_{n,1}^*, \dots, b_{n,\kappa_n}^*) = \#(\text{Cut}(\mathbf{t}_n), \varphi_n),$$

and the claim follows readily from Lemma 2. □

To complete the proof of Theorem 3, we need to identify the limiting distribution $\#(\mathcal{T}, \Phi)$. For this, we use a second discrete approximation of the latter. Denote by \mathbb{T}_n a Galton–Watson tree with Poisson(1) offspring distribution conditioned to have n vertices and where labels are assigned to the vertices uniformly at random. It is known, see for instance Aldous [1], that \mathbb{T}_n is distributed as a uniform rooted Cayley tree with n vertices and that, reduced to k vertices picked uniformly at random and rescaled by a factor $n^{-1/2}$, it converges to the CRT reduced to k leaves. We endow \mathbb{T}_n with the marking–erasing process ψ_n associated with the process which marks each vertex independently with probability p_n . Adapting Lemma 2 to \mathbb{T}_n and the uniform probability on vertices, we get

$$\lim_{n \rightarrow \infty} n^{-1} \#(\mathbb{T}_n, \psi_n) = \#(\mathcal{T}, \Phi) \quad \text{in distribution,} \tag{5}$$

where $\#(\mathbb{T}_n, \psi_n)$ stands here for the number of vertices of each component, the root-component first, and the next in non-increasing order. We now study the asymptotic behavior of this vector in order to show that the right-hand side above is the limit in Theorem 3.

3.3. Asymptotic behavior of the size of the burnt blocks

Denote by $C_{n,0}$ the size of the connected component of \mathbb{T}_n that contains the root, M_n the number of marks, and $C_{n,1}^* \geq \dots \geq C_{n,M_n}^*$ the respective sizes of the other connected components, listed in non-increasing order.

Proposition 3. *For all continuous and bounded maps $f : (0, 1) \rightarrow \mathbf{R}$ and $F : \ell^1(\mathbf{R}) \rightarrow \mathbf{R}$,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbf{E} \left[f \left(\frac{C_{n,0}}{n} \right) F \left(\frac{C_{n,1}^*}{n}, \dots, \frac{C_{n,M_n}^*}{n} \right) \right] \\ &= \int_0^1 f(x) \mathbf{E} \left[F \left(\frac{(1-x)J_1}{\sigma(1)}, \frac{(1-x)J_2}{\sigma(1)}, \dots \right) \middle| \sigma(1) = \frac{1-x}{c^2 x^2} \right] \mathbf{P}(D(c) \in dx), \end{aligned}$$

where σ is a subordinator distributed as (3) and $\mathbf{P}(D(c) \in dx)$ is defined in (1).

Before proving this result, notice first that Theorem 3 is a direct consequence of Propositions 2 and 3 and the convergence (5).

Proof of Theorem 3. Let $f : (0, 1) \rightarrow \mathbf{R}$ and $F : \ell^1(\mathbf{R}) \rightarrow \mathbf{R}$ be two continuous and bounded maps. From Proposition 2 and (5), the sequences

$$\mathbb{E} \left[f \left(\frac{I_n}{n} \right) F \left(\frac{b_{n,1}^*}{n}, \dots, \frac{b_{n,k_n}^*}{n} \right) \right] \quad \text{and} \quad \mathbb{E} \left[f \left(\frac{C_{n,0}}{n} \right) F \left(\frac{C_{n,1}^*}{n}, \dots, \frac{C_{n,M_n}^*}{n} \right) \right]$$

both converge to the same limit as $n \rightarrow \infty$ and Proposition 3 gives the expression of the latter, which is the one claimed in Theorem 3. □

It remains to prove Proposition 3. For any positive real number z , we define the Borel distribution with parameter z , which is the law of the size of a Galton–Watson tree with $\text{Poisson}(z)$ offspring distribution:

$$\mathbb{P}(\text{Borel}(z) = n) = \frac{1}{n!} e^{-nz} (nz)^{n-1}, \quad n \geq 1.$$

We also define for any integer k , the Borel–Tanner distribution with parameter k as the sum of k i.i.d. Borel(1) variables:

$$\mathbb{P}(\text{Borel–Tanner}(k) = n) = \frac{k}{(n-k)!} e^{-n} n^{n-k-1}, \quad n \geq k.$$

Borel and Borel–Tanner distributions appear in our context as the sizes of the connected components of \mathbb{T}_n .

Lemma 3. *For any integers x, y with $x + y \leq n$, conditionally on the event $\{C_{n,0} = x, M_n = y\}$, the vector $(C_{n,1}^*, \dots, C_{n,y}^*)$ is distributed as a non-increasing rearrangement of y i.i.d. Borel(1) random variables conditioned to have sum $n - x$.*

Proof. We explicitly write the condition for the size of the tree. Let \mathbb{T} be a Galton–Watson tree with $\text{Poisson}(1)$ offspring distribution; we endow it with the marking–erasing process associated with the process which marks each vertex independently with probability p_n . Denote by \tilde{M}_n the number of marks, $\tilde{C}_{n,0}$ the size of the root-component and, conditionally on $\{\tilde{M}_n = y\}$, $\tilde{C}_{n,1}^* \geq \dots \geq \tilde{C}_{n,y}^*$ the ranked sizes of the other components. Note that on the event $\{\tilde{M}_n = y\}$, we have $|\mathbb{T}| = \tilde{C}_{n,0} + \tilde{C}_{n,1}^* + \dots + \tilde{C}_{n,y}^*$.

Condition on the event $\{\tilde{M}_n = y\}$; it is known that the subtrees of \mathbb{T} generated by the y atoms of the point process are independent Galton–Watson trees with $\text{Poisson}(1)$ offspring distribution, independent of $\tilde{C}_{n,0}$. Hence, on the event $\{\tilde{M}_n = y\}$, $\tilde{C}_{n,1}^*, \dots, \tilde{C}_{n,y}^*$ are i.i.d. Borel(1) random variables, listed in non-increasing order and independent of $\tilde{C}_{n,0}$. Further, on the event $\{|\mathbb{T}| = n, \tilde{M}_n = y, \tilde{C}_{n,0} = x\}$, $\tilde{C}_{n,1}^*, \dots, \tilde{C}_{n,y}^*$ are conditioned to have sum $n - x$. □

The Borel(1) distribution belongs to the domain of attraction of a stable law of index $1/2$. A consequence tailored for our need is the following: let $(\beta_i)_{i \geq 1}$ be i.i.d. Borel(1) random variables, and for any $k \geq 1$, denote by $\beta_1^* \geq \dots \geq \beta_k^*$ the order statistics of the first k elements of the latter. Let also σ be a subordinator distributed as (3) and $J_1 \geq J_2 \geq \dots \geq 0$ the ranked sizes of its jumps made during the time interval $[0, 1]$.

Lemma 4. *Let $\lambda, \nu > 0$ and two sequences of integers k_n and a_n such that $\lim_{n \rightarrow \infty} n^{-1/2} k_n = \lambda$ and $\lim_{n \rightarrow \infty} n^{-1} \times a_n = \nu$. Then*

$$\lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \sum_{i=1}^{\lfloor \sqrt{nt} \wedge k_n \rfloor} \beta_i, t \geq 0 \right) \middle| \sum_{i=1}^{k_n} \beta_i = a_n \right) = ((\sigma(t \wedge \lambda), t \geq 0) | \sigma(\lambda) = \nu)$$

in distribution for the Skorohod topology. As a consequence, the convergence of the ranked jumps holds for the ℓ^1 topology:

$$\lim_{n \rightarrow \infty} \left(\left(\frac{\beta_1^*}{n}, \dots, \frac{\beta_{k_n}^*}{n} \right) \middle| \sum_{i=1}^{k_n} \beta_i = a_n \right) = \left(\frac{vJ_1}{\sigma(1)}, \frac{vJ_2}{\sigma(1)}, \dots \middle| \sigma(1) = \frac{v}{\lambda^2} \right)$$

in distribution.

Proof. The first convergence is the result stated in Lemma 11 of Aldous and Pitman [2]. The second then follows from the continuity obtained in Lemma 1. \square

We apply this convergence to the random sequences M_n and $n - C_{n,0}$ instead of k_n and a_n . They fulfill the assumptions of Lemma 4 as it is shown in the following lemma that we prove in the next subsection.

Lemma 5. *Let $D(c)$ be a random variable distributed as (1). Then*

$$\lim_{n \rightarrow \infty} \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) = (D(c), cD(c)) \quad \text{in distribution.}$$

In order to go from deterministic sequences to random sequences, we also use the following elementary result (see Carathéodory [6], Part Four, Chapter I). Let \mathbb{X} and \mathbb{Y} be metric spaces and f, f_1, f_2, \dots be functions defined from \mathbb{X} to \mathbb{Y} . We say that f_n converges continuously to f if for any $x, x_1, x_2, \dots \in \mathbb{X}$ such that $\lim_{n \rightarrow \infty} x_n = x$ in \mathbb{X} , we have $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ in \mathbb{Y} . Then f_n converges continuously to f if and only if f is continuous and f_n converges to f uniformly on compact sets.

Proof of Proposition 3. Let $f : (0, 1) \rightarrow \mathbf{R}$ and $F : \ell^1(\mathbf{R}) \rightarrow \mathbf{R}$ be two continuous and bounded maps. With the notations of Lemma 4, define for any $(u, v) \in (0, 1) \times (0, \infty)$

$$\Upsilon_n(u, v) := f(u) \mathbb{E} \left[F \left(\frac{\beta_1^*}{n}, \dots, \frac{\beta_{\lfloor \sqrt{nv} \rfloor}^*}{n} \right) \middle| \sum_{i=1}^{\lfloor \sqrt{nv} \rfloor} \beta_i = n - \lfloor nu \rfloor \right]$$

and

$$\Upsilon(u, v) := f(u) \mathbb{E} \left[F \left(\frac{(1-u)J_1}{\sigma(1)}, \frac{(1-u)J_2}{\sigma(1)}, \dots \right) \middle| \sigma(1) = \frac{1-u}{v^2} \right].$$

Then Lemma 4 states that $\Upsilon_n(u_n, v_n) \rightarrow \Upsilon(u, v)$ whenever $(u_n, v_n) \rightarrow (u, v)$. On the one hand, $\mathbb{E}[\Upsilon(D(c), cD(c))]$ is the limit claimed in Proposition 3 and, from Lemma 3, the $C_{n,i}^*$'s are, conditionally given $C_{n,0}$ and M_n , distributed as ranked i.i.d. Borel(1) random variables conditioned to have sum $n - C_{n,0}$. Then we also have

$$\mathbb{E} \left[\Upsilon_n \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) \right] = \mathbb{E} \left[f \left(\frac{C_{n,0}}{n} \right) F \left(\frac{C_{n,1}^*}{n}, \dots, \frac{C_{n,M_n}^*}{n} \right) \right].$$

On the other hand, from the discussion above, Υ is continuous (which is also a consequence of Proposition 1) and $\Upsilon_n \rightarrow \Upsilon$ uniformly on compact sets. Let us bound from above

$$\left| \mathbb{E} \left[\Upsilon_n \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) \right] - \mathbb{E}[\Upsilon(D(c), cD(c))] \right|$$

by

$$\mathbb{E} \left[\left| \Upsilon_n \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) - \Upsilon \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) \right| \right] + \left| \mathbb{E} \left[\Upsilon \left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) \right] - \mathbb{E}[\Upsilon(D(c), cD(c))] \right|.$$

From Lemma 5, since Υ is continuous and bounded, the second term tends to 0. Moreover $\Upsilon, \Upsilon_1, \Upsilon_2, \dots$ are uniformly bounded, say by $C > 0$, therefore the first term is bounded from above by

$$\sup_{x \in K} |\Upsilon_n(x) - \Upsilon(x)| + 2CP \left(\left(\frac{C_{n,0}}{n}, \frac{M_n}{\sqrt{n}} \right) \notin K \right),$$

for any compact K . The first term of the latter converges to 0 for any K and the second can be made arbitrary small as the sequence is tight. \square

3.4. Asymptotic behavior of the number of burnt blocks

We finally prove Lemma 5 which completes the proof of Proposition 3 and thereby that of Theorem 3. We use the two following observations: the convergence of the first marginal $n^{-1}C_{n,0}$ holds and the conditional distribution of M_n given $C_{n,0}$ is known explicitly.

Lemma 6. *We have $\lim_{n \rightarrow \infty} n^{-1}C_{n,0} = D(c)$ in distribution.*

Proof. Denote by $\mu(\xi_0)$ the mass of the root-component of the CRT after logging at the atoms of a Poisson point process with rate c per unit length (here keeping only the closest atoms to the root does not matter). Then (5) yields $\lim_{n \rightarrow \infty} n^{-1}C_{n,0} = \mu(\xi_0)$ in distribution. The claim follows from the identity $\mu(\xi_0) = D(c)$ in distribution stated in Corollary 5 of Aldous and Pitman [3] since $\mu(\xi_0)$ here is $Y_1^*(c)$ there. \square

Lemma 7. *For any $n \geq 2$, the pair $(C_{n,0}, M_n)$ is distributed as follows: for any integers x, y such that $x + y \leq n$,*

$$P(C_{n,0} = x, M_n = y) = \frac{n!(x(1 - p_n))^{x-1}(xp_n)^y(n - x)^{n-x-y-1}}{n^{n-1}x!(y-1)!(n-x-y)!}.$$

Then, on the event $\{C_{n,0} = x\}$, M_n is distributed as $X_n + 1$ where X_n is a binomial random variable with parameters $n - x - 1$ and $(xp_n)/(n - x + xp_n)$.

Proof. For the first claim, as in the proof of Lemma 3, we explicitly write the condition on the size of the tree and work with a Galton–Watson tree with Poisson(1) offspring distribution \mathbb{T} :

$$\begin{aligned} P(C_{n,0} = x, M_n = y | |\mathbb{T}| = n) \\ = \frac{P(C_{n,0} = x)P(M_n = y | C_{n,0} = x)P(|\mathbb{T}| = n | C_{n,0} = x, M_n = y)}{P(|\mathbb{T}| = n)}. \end{aligned}$$

We know that $|\mathbb{T}|$ is Borel(1) distributed. Moreover, the root-component of \mathbb{T} is a Galton–Watson tree with Poisson($1 - p_n$) offspring distribution, so that $C_{n,0}$ is Borel($1 - p_n$) distributed, and on $\{C_{n,0} = x\}$, M_n is the sum of x i.i.d. Poisson(p_n) random variables, so is Poisson(xp_n) distributed. Finally, from Lemma 3, on $\{C_{n,0} = x, M_n = y\}$, $|\mathbb{T}| - x$ is the sum of y i.i.d. Borel(1) random variables, i.e. is Borel–Tanner(y) distributed. Putting the pieces together gives the first claim. For the second claim (with the implicit condition $|\mathbb{T}| = n$), we then directly compute

$$\begin{aligned} P(M_n = y | C_{n,0} = x) &= P(C_{n,0} = x, M_n = y) \left(\sum_{z=1}^{n-x} P(C_{n,0} = x, M_n = z) \right)^{-1} \\ &= \frac{(n-x-1)!}{(y-1)!(n-x-y)!} \left(\frac{xp_n}{n-x+xp_n} \right)^{y-1} \left(\frac{n-x}{n-x+xp_n} \right)^{n-x-y} \\ &= P(X_n = y - 1), \end{aligned}$$

where X_n is the desired binomial random variable. \square

We can now prove Lemma 5.

Proof of Lemma 5. We aim to show that for any $s, t \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left(-s \frac{C_{n,0}}{n} - t \frac{M_n}{\sqrt{n}} \right) \right] = \mathbb{E} \left[\exp \left(-(s + ct) D(c) \right) \right].$$

From Lemma 6, $n^{-1} C_{n,0} \rightarrow D(c)$ in distribution, it is thus sufficient to show

$$\limsup_{n \rightarrow \infty} \left| \mathbb{E} \left[\exp \left(-s \frac{C_{n,0}}{n} - t \frac{M_n}{\sqrt{n}} \right) \right] - \mathbb{E} \left[\exp \left(-(s + ct) \frac{C_{n,0}}{n} \right) \right] \right| = 0.$$

Let $\varepsilon > 0$ and fix $\delta > 0$ such that for any n large enough, $\mathbb{P}(C_{n,0} > (1 - \delta)n) \leq \varepsilon$. We then reduce to show the above convergence on the event $\{C_{n,0} \leq (1 - \delta)n\}$. Using Lemma 7, we compute for any $1 \leq x \leq n - 1$ and any $t \geq 0$,

$$\mathbb{E}[e^{-tM_n} | C_{n,0} = x] = \mathbb{E}[e^{-t(X_n+1)}] = e^{-t} \left(1 - \frac{xp_n(1 - e^{-t})}{n - x + xp_n} \right)^{n-x-1}.$$

Conditioning first on the value of $C_{n,0}$ and then averaging, we obtain

$$\begin{aligned} & \mathbb{E} \left[\exp \left(-s \frac{C_{n,0}}{n} - t \frac{M_n}{\sqrt{n}} \right) \mathbf{1}_{\{C_{n,0} \leq (1-\delta)n\}} \right] \\ &= \sum_{x=1}^{\lfloor (1-\delta)n \rfloor} \mathbb{P}(C_{n,0} = x) \exp \left(-\frac{sx}{n} \right) \exp \left(-\frac{t}{\sqrt{n}} \right) \left(1 - \frac{xp_n(1 - e^{-t/\sqrt{n}})}{n - x + xp_n} \right)^{n-x-1}. \end{aligned}$$

Remark that, uniformly for $x \leq \lfloor (1 - \delta)n \rfloor$,

$$\frac{xp_n(1 - e^{-t/\sqrt{n}})}{n - x + xp_n} = \frac{1}{n - x} \left(\frac{xct}{n} + o(1) \right) \quad \text{as } n \rightarrow \infty.$$

As a consequence, as $n \rightarrow \infty$,

$$\exp \left(-\frac{t}{\sqrt{n}} \right) \left(1 - \frac{xp_n(1 - e^{-t/\sqrt{n}})}{n - x + xp_n} \right)^{n-x-1} = \exp \left(-\frac{xct}{n} \right) (1 + o(1)),$$

uniformly for $x \leq \lfloor (1 - \delta)n \rfloor$. Finally, the difference

$$\mathbb{E} \left[\exp \left(-s \frac{C_{n,0}}{n} - t \frac{M_n}{\sqrt{n}} \right) \mathbf{1}_{\{C_{n,0} \leq (1-\delta)n\}} \right] - \mathbb{E} \left[\exp \left(-(s + ct) \frac{C_{n,0}}{n} \right) \mathbf{1}_{\{C_{n,0} \leq (1-\delta)n\}} \right]$$

tends to 0 as $n \rightarrow \infty$, which completes the proof. □

4. Asymptotic proportion of fireproof vertices in the subcritical regime

We now consider the subcritical regime $p_n \gg n^{-1/2}$ of the dynamics on \mathbf{t}_n . We prove the convergence of the total number of fireproof vertices I_n , rescaled by a factor p_n^2 (recall Theorem 1), and also the following result on the size of the largest fireproof component.

Proposition 4. *For any $\varepsilon > 0$, with a probability converging to 1 as $n \rightarrow \infty$, there exists at least one fireproof subtree larger than $n^{-\varepsilon} p_n^{-2}$ but none larger than εp_n^{-2} .*

Let us sketch our approach to establish Theorem 1. We let the dynamics evolve until an edge is set on fire for the first time, denoting this random time by $\zeta_n \in \mathbf{N} \cup \{\infty\}$. The event $\{\zeta_n = \infty\}$ corresponds to the case where the whole tree is fireproof at the end. Conditionally on $\{\zeta_n = k\}$ with $k \in \mathbf{N}$, if we delete the $k - 1$ first fireproof edges, we get a decomposition of \mathbf{t}_n into a forest of k trees. Then we set on fire an edge of this forest uniformly at random and burn the whole subtree that contains the latter. The burnt subtree is therefore picked at random with a probability proportional to its number of edges. We then study the dynamics which continue independently on each of the $k - 1$ other subtrees.

Let σ be a subordinator distributed as (3) and $J_1 \geq J_2 \geq \dots \geq 0$ the sizes of its jumps made during the time interval $[0, 1]$. Let also \mathbf{e} be an exponential random variable with parameter 1 independent of σ . We shall see that $p_n \zeta_n$ converges to \mathbf{e} in distribution and that the sequence of the sizes of the non-burnt subtrees at time ζ_n , ranked in non-increasing order and rescaled by a factor p_n^2 , converges in distribution to $(\mathbf{e}^2 J_k)_{k \geq 1}$ in ℓ^1 . Conditionally given $(\mathbf{e}^2 J_k)_{k \geq 1}$, we define a sequence $(X_k(\mathbf{e}))_{k \geq 1}$ of independent random variables sampled according to $\mu_{\mathbf{e}^2 J_k}$ respectively, where for every $x > 0$, μ_x is the probability measure given by

$$\mu_x(dy) = \left(\frac{x^3}{2\pi y(x-y)^3}\right)^{1/2} \exp\left(-\frac{xy}{2(x-y)}\right) dy, \quad 0 < y < x. \tag{6}$$

Note that if X is distributed as μ_x , then $x^{-1}X$ is distributed as $D(x^{1/2})$, defined in (1). Indeed, μ_x is the limit of the number of fireproof vertices in a subtree of asymptotic size x (see Lemma 8 for a precise statement). Informally, summing over all subtrees, since the dynamics on each are independent, we get

$$\lim_{n \rightarrow \infty} p_n^2 I_n = \sum_{k=1}^{\infty} X_k(\mathbf{e}) \quad \text{in distribution.} \tag{7}$$

Theorem 1 finally follows from the identity

$$\sum_{k=1}^{\infty} X_k(\mathbf{e}) = Z^2 \quad \text{in distribution.} \tag{8}$$

To derive the latter, note that, conditionally given \mathbf{e} , the sequence $(\mathbf{e}^2 J_k)_{k \geq 1}$ is distributed as the ranked atoms of a Poisson random measure on $(0, \infty)$ with intensity $\mathbf{e}(2\pi x^3)^{-1/2} dx$. Further, conditionally given \mathbf{e} , the sequence $(\mathbf{e}^2 J_k, X_k(\mathbf{e}))_{k \geq 1}$ is distributed as the atoms of a Poisson random measure on $(0, \infty)^2$ with intensity $\mathbf{e}(2\pi x^3)^{-1/2} dx \mu_x(dy)$, ranked in the non-increasing order of the first coordinate. Therefore, conditioning first on \mathbf{e} , using Laplace formula and then averaging, we have for any $q > 0$,

$$\mathbf{E} \left[\exp\left(-q \sum_{k=1}^{\infty} X_k(\mathbf{e})\right) \right] = \int_0^{\infty} \exp\left(-\int_{(0,\infty)^2} (1 - e^{-qy}) \frac{t}{\sqrt{2\pi x^3}} dx \mu_x(dy)\right) e^{-t} dt.$$

Using the definition of μ_x and the change of variables $(x, y) \mapsto (y(x - y)^{-1/2}, y)$, we see that the right-hand side is equal to

$$\int_0^{\infty} \exp\left(-t - t \int_0^{\infty} (1 - e^{-qy}) e^{-y/2} \frac{dy}{\sqrt{2\pi y^3}}\right) dt.$$

We write

$$(1 - e^{-qy}) e^{-y/2} = \left(1 - \exp\left(-\frac{2q+1}{2}y\right)\right) - \left(1 - \exp\left(-\frac{y}{2}\right)\right);$$

since

$$\int_0^{\infty} \left(1 - \exp\left(-\frac{z^2 y}{2}\right)\right) \frac{dy}{\sqrt{2\pi y^3}} = z \quad \text{for any } z > 0,$$

we finally obtain,

$$\mathbb{E} \left[\exp \left(-q \sum_{k=1}^{\infty} X_k(\mathbf{e}) \right) \right] = \int_0^{\infty} \exp(-t\sqrt{2q+1}) dt = (2q+1)^{-1/2} = \mathbb{E}[\exp(-qZ^2)].$$

In the rest of this section, we first prove the convergence of the sequence of the sizes of the non-burnt trees after the first fire. We then establish (7) which, by (8), proves Theorem 1. Finally, we prove Proposition 4.

4.1. Configuration at the instant of the first fire

Recall that we denote by ζ_n the first instant where an edge is set on fire during the dynamics on \mathbf{t}_n . Then ζ_n is a truncated geometric random variable:

$$\begin{aligned} \mathbb{P}(\zeta_n = \infty) &= (1 - p_n)^{n-1} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{and} \\ \mathbb{P}(\zeta_n = k) &= p_n(1 - p_n)^{k-1} \quad \text{for every } k \in \{1, \dots, n-1\}, \end{aligned}$$

and $p_n\zeta_n$ converges in distribution to an exponential random variable with parameter 1. For each integer $k \leq n$, we denote by $\mathbf{t}_{n,1}, \dots, \mathbf{t}_{n,k}$ the forest obtained by deleting $k-1$ edges of \mathbf{t}_n uniformly at random, where the labeling is made uniformly at random, and $|\mathbf{t}_{n,1}|^* \geq \dots \geq |\mathbf{t}_{n,k}|^*$ a non-increasing rearrangement of their sizes. We know from Lemma 5 in Bertoin [5] (see also Pavlov [8] or Pitman [10]) that the sizes of these k subtrees are distributed as i.i.d. Borel(1) random variables conditioned to have sum n : for any $n_1, \dots, n_k \geq 1$ such that $n_1 + \dots + n_k = n$,

$$\mathbb{P}(|\mathbf{t}_{n,1}| = n_1, \dots, |\mathbf{t}_{n,k}| = n_k) = \frac{(n-k)!}{kn^{n-k-1}} \prod_{j=1}^k \frac{n_j^{n_j-1}}{n_j!}. \tag{9}$$

Moreover, conditionally on the partition of $\{1, \dots, n\}$ induced by the k subsets of vertices of these subtrees, the $\mathbf{t}_{n,i}$'s are independent uniform Cayley trees on their respective set of vertices.

Notice that on the event $\{\zeta_n \geq k\}$, the forest obtained by deleting the first $k-1$ fireproof edges is distributed as $\mathbf{t}_{n,1}, \dots, \mathbf{t}_{n,k}$ and, further, ζ_n is independent of the latter. The forest at the instant of the first fire is then distributed as follows: we first sample a (truncated) geometric variable ζ_n and then independently the uniform forest $\mathbf{t}_{n,1}, \dots, \mathbf{t}_{n,\zeta_n}$. Recall that the Borel(1) distribution belongs to the domain of attraction of a stable law of index $1/2$ so that, taking a number of order p_n^{-1} of i.i.d. such random variables, the sum is typically of order p_n^{-2} . Then, loosely speaking, conditioning this sum to be abnormally large, here of order n , essentially amounts to conditioning one single variable to be large, the others being almost unaffected. This feature is formalized in the next proposition.

Proposition 5. *We have*

$$\lim_{n \rightarrow \infty} p_n^2 (n - |\mathbf{t}_{n,1}|^*, |\mathbf{t}_{n,2}|^*, \dots, |\mathbf{t}_{n,\zeta_n}|^*) = \mathbf{e}^2(\sigma(1), J_1, J_2, \dots) \quad \text{in distribution.}$$

Proof. The proof is similar to that of Proposition 3. Aldous and Pitman [3], equation (34), provide the convergence in distribution

$$\lim_{n \rightarrow \infty} k_n^{-2} (n - |\mathbf{t}_{n,1}|^*, |\mathbf{t}_{n,2}|^*, \dots, |\mathbf{t}_{n,k_n}|^*) = (\sigma(1), J_1, J_2, \dots)$$

for any sequence $k_n = o(n^{1/2})$. Let $f : \ell^1(\mathbf{R}) \rightarrow \mathbf{R}$ be a continuous and bounded function and set for any $x > 0$

$$F_n(x) := \mathbb{E} \left[f \left(p_n^2 (n - |\mathbf{t}_{n,1}|^*, |\mathbf{t}_{n,2}|^*, \dots, |\mathbf{t}_{n,\lfloor xp_n^{-1} \rfloor}|^*) \right) \right],$$

and

$$F(x) := \mathbb{E} \left[f \left(x^2 (\sigma(1), J_1, J_2, \dots) \right) \right].$$

The previous convergence yields $\lim_{n \rightarrow \infty} F_n(x_n) = F(x)$ whenever $\lim_{n \rightarrow \infty} x_n = x$. Using Skorohod’s representation theorem, we may suppose $\lim_{n \rightarrow \infty} p_n \zeta_n = \mathbf{e}$ almost surely. Since ζ_n is independent of the $\mathbf{t}_{n,i}$ ’s, we have $\lim_{n \rightarrow \infty} F_n(p_n \zeta_n) = F(\mathbf{e})$ almost surely and the claim follows from Lebesgue’s theorem. \square

Recall that the first fire burns one subtree in the forest $\mathbf{t}_{n,1}, \dots, \mathbf{t}_{n,\zeta_n}$ and that the latter is chosen at random with a probability proportional to its size minus one. Therefore, with high probability, this burnt subtree has a size of order n and the forest that we obtain by discarding this tree and the edges previously fireproof has a total size of order $p_n^{-2} = o(n)$. This already strengthens the result (i) of the Introduction. The fire dynamics then continue independently on each tree of this forest and the total number of fireproof vertices is the sum of the number of fireproof vertices in each component.

4.2. Total number of fireproof vertices

We now study the dynamics on the remaining forest after the first fire. We know from Proposition 5 that with high probability, the largest trees have size of order p_n^{-2} so that they are now critical for the dynamics which continue on each with parameter $p_n = (p_n^{-2})^{-1/2}$. To see this, we slightly generalize the convergence (iii) of the Introduction. Let $(\mathbf{t}'_n)_{n \geq 1}$ be a sequence of Cayley trees with size $|\mathbf{t}'_n| \sim ap_n^{-2}$ as $n \rightarrow \infty$ for some $a > 0$ and define $I'_n = \text{Card}\{i \in \mathbf{t}'_n : i \text{ is fireproof}\}$.

Lemma 8. *The law of $p_n^2 I'_n$ converges weakly to the distribution μ_a defined by (6).*

Proof. The proof of Theorem 1 in Bertoin [5] shows that $|\mathbf{t}'_n|^{-1} I'_n$, the proportion of fireproof vertices in \mathbf{t}'_n , converges in distribution to $D(a^{1/2})$, as defined in (1). Since $p_n^2 |\mathbf{t}'_n| \rightarrow a$, we get $p_n^2 I'_n \rightarrow aD(a^{1/2})$ in distribution. One easily checks that the latter is distributed according to μ_a . \square

Using Proposition 5 and Lemma 8, we can now prove (7) and so, Theorem 1.

Proof of Theorem 1. Conditionally given ζ_n , we write $(\mathbf{t}'_{n,1}, \dots, \mathbf{t}'_{n,\zeta_n})$ for the trees obtained by deleting the first $\zeta_n - 1$ fireproof edges, listed so that $\mathbf{t}'_{n,1}$ is the tree burnt at time ζ_n and $|\mathbf{t}'_{n,2}| \geq \dots \geq |\mathbf{t}'_{n,\zeta_n}|$. Note that

$$I_n = \sum_{k=2}^{\zeta_n} \text{Card}\{i \in \mathbf{t}'_{n,k} : i \text{ is fireproof}\}.$$

From Proposition 5, we have

$$\lim_{n \rightarrow \infty} p_n^2 (n - |\mathbf{t}'_{n,1}|, |\mathbf{t}'_{n,2}|, \dots, |\mathbf{t}'_{n,\zeta_n}|) = \mathbf{e}^2(\sigma(1), J_1, J_2, \dots)$$

in distribution. Therefore, for any $\varepsilon > 0$ there exists $N \in \mathbf{N}$ and then $n_0 \in \mathbf{N}$ such that

$$\mathbb{P}\left(\sum_{k=N}^{\infty} \mathbf{e}^2 J_k > \varepsilon\right) < \varepsilon \quad \text{and} \quad \text{for any } n \geq n_0, \quad \mathbb{P}\left(\sum_{k=N+1}^{\zeta_n} p_n^2 |\mathbf{t}'_{n,k}| > \varepsilon\right) < \varepsilon.$$

Recall that, conditionally given $(\mathbf{e}^2 J_k)_{k \geq 1}, (X_k(\mathbf{e}))_{k \geq 1}$ is a sequence of independent random variables sampled according to $\mu_{\mathbf{e}^2 J_k}$ respectively, where for every $x > 0$, μ_x is the probability measure on $(0, x)$ given by (6). In particular, $X_k(\mathbf{e}) \leq \mathbf{e}^2 J_k$ for every $k \geq 1$; we also have $\text{Card}\{i \in \mathbf{t}'_{n,k} : i \text{ is fireproof}\} \leq |\mathbf{t}'_{n,k}|$. Then

$$\mathbb{P}\left(\sum_{k=N}^{\infty} X_k(\mathbf{e}) > \varepsilon\right) < \varepsilon,$$

and for any $n \geq n_0$,

$$P\left(\sum_{k=N+1}^{\xi_n} p_n^2 \text{Card}\{i \in \mathbf{t}'_{n,k} : i \text{ is fireproof}\} > \varepsilon\right) < \varepsilon.$$

Conditionally on the partition of $\{1, \dots, n\}$ induced by the subsets of vertices of the subtrees, the $\mathbf{t}'_{n,k}$'s are independent uniform Cayley trees on their respective set of vertices. Proposition 5 and Lemma 8 thus yield

$$\lim_{n \rightarrow \infty} \sum_{k=2}^N p_n^2 \text{Card}\{i \in \mathbf{t}'_{n,k} : i \text{ is fireproof}\} = \sum_{k=1}^{N-1} X_k(\mathbf{e}) \quad \text{in distribution.}$$

Since the rests are arbitrary small with high probability, we get

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{\xi_n} p_n^2 \text{Card}\{i \in \mathbf{t}'_{n,k} : i \text{ is fireproof}\} = \sum_{k=1}^{\infty} X_k(\mathbf{e}) \quad \text{in distribution.}$$

The above convergence is (7), Theorem 1 then follows from (8). □

Combined with the results of Bertoin [5], Proposition 5 and Lemma 8 also entail Proposition 4 about the size of the largest fireproof connected component.

Proof of Proposition 4. Fix $\varepsilon > 0$ and $\delta \in (0, 1/2)$. Let σ be a subordinator distributed as (3) and $\chi \in (0, \varepsilon)$ such that the probability that σ admits no jump larger than χ during the time interval $[0, 1]$ is less than δ . Consider the subtrees of \mathbf{t}_n larger than χp_n^{-2} when an edge is set on fire for the first time. From Proposition 5, we know that the number of such trees converges to the number of jumps larger than χ made by σ before time 1. The latter is almost surely finite and non-zero with a probability greater than $1 - \delta$. Now from Lemma 8, these subtrees are critical and thus for each, from Corollary 1 and Proposition 1 of Bertoin [5], the probability that there exists a fireproof component larger than εp_n^{-2} tends to 0 and the probability that there exists at least one larger than $n^{-\varepsilon} p_n^{-2}$ tends to 1. Therefore for any n large enough, on the one hand there exists in \mathbf{t}_n a fireproof subtree larger than $n^{-\varepsilon} p_n^{-2}$ and on the other hand there exists none larger than εp_n^{-2} , both with a probability at least $1 - 2\delta$. The claim follows since δ is arbitrary. □

5. Asymptotic proportion of burnt vertices in the supercritical regime

We finally consider the supercritical regime $n^{-1} \ll p_n \ll n^{-1/2}$ and prove Theorem 2. Recall that $b_{n,1}, \dots, b_{n,\kappa_n}$ denote the sizes of the burnt subtrees, listed in order of appearance. Let $(\mathbf{e}_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter 1 and for each $i \geq 1$, denote by $\gamma_i := \mathbf{e}_1 + \dots + \mathbf{e}_i$. Let also $(Z_i)_{i \geq 1}$ be a sequence of i.i.d. standard Gaussian random variables, independent of $(\gamma_i)_{i \geq 1}$. We shall prove the following result.

Theorem 4. *We have*

$$\lim_{n \rightarrow \infty} (np_n)^{-2} (b_{n,1}, \dots, b_{n,\kappa_n}) = (\gamma_i^{-2} Z_i^2)_{i \geq 1}$$

in distribution for the ℓ^1 topology.

Theorem 2 follows as a corollary.

Proof of Theorem 2. As a consequence of Theorem 4, we have the convergence of the sums:

$$\lim_{n \rightarrow \infty} (np_n)^{-2} B_n = \sum_{i=1}^{\infty} \gamma_i^{-2} Z_i^2 \quad \text{in distribution.}$$

Note that the sequence $(\gamma_i)_{i \geq 1}$ is distributed as the atoms of a Poisson random measure on $(0, \infty)$ with intensity dx , it follows readily that the sequence $(\gamma_i^{-2} Z_i^2)_{i \geq 1}$ is distributed as the atoms of a Poisson random measure on $(0, \infty)$ with intensity $(2\pi x^3)^{-1/2} dx$. The above limit is thus distributed as $\sigma(1)$ where σ is the subordinator defined by (3); Theorem 2 finally follows from the well-known identity $\sigma(1) = Z^{-2}$ in distribution. \square

As discussed in the Introduction, in order to prove Theorem 4, we first show the joint convergence of the first j coordinates for any $j \geq 1$, and then that, taking j large enough, the other coordinates are arbitrary small with high probability. We conclude in the same manner as in the proof of Theorem 1.

5.1. Asymptotic size of the first burnt subtrees

We first prove the convergence of the size of the first burnt subtree $b_{n,1}$. As in the preceding section, we let the dynamics evolve until an edge is set on fire for the first time, denoting this random time by ζ_n . The size of the tree that burns at this instant is distributed as one among ζ_n i.i.d. Borel(1) random variables conditioned to have sum n , chosen proportionally to its value minus 1. As we have seen, $p_n \zeta_n$ converges in distribution to an exponential random variable with parameter 1, thus ζ_n is typically of order p_n^{-1} and the sum of ζ_n i.i.d. Borel(1) random variables is of order p_n^{-2} . In the previous section, we considered $p_n^{-2} = o(n)$ and we have seen that conditioning these random variables to have sum n essentially amounts to conditioning one to be of order n (Proposition 5). The behavior is notoriously different when $n = o(p_n^{-2})$. As an example, Pavlov [8], Theorem 3, gives an asymptotic of the size of the largest subtree when one removes $k_n - 1$ edges uniformly at random, with $n = o(k_n^2)$.

Lemma 9. *As $n \rightarrow \infty$, $(np_n)^{-2} b_{n,1}$ converges in distribution to $e^{-2} Z^2$ where Z and e are independent, respectively standard Gaussian and exponential with parameter 1 distributed.*

Proof. We work throughout the proof conditionally on $\{\zeta_n = k_n\}$ with $k_n \sim cp_n^{-1}$, $c > 0$ arbitrary and show the convergence in distribution $(np_n)^{-2} b_{n,1} \rightarrow c^{-2} Z^2$. The general claim then follows as in the proof of Proposition 5. For any $\lambda \geq 0$, we write

$$\begin{aligned} \mathbb{E}[e^{-\lambda(np_n)^{-2} b_{n,1}} | \zeta_n = k_n] &= \sum_{m=0}^{\infty} e^{-\lambda(np_n)^{-2} m} \mathbb{P}(b_{n,1} = m | \zeta_n = k_n) \\ &= \int_0^{\infty} e^{-\lambda(np_n)^{-2} \lfloor x \rfloor} \mathbb{P}(b_{n,1} = \lfloor x \rfloor | \zeta_n = k_n) dx \\ &= \int_0^{\infty} e^{-\lambda(np_n)^{-2} \lfloor x(np_n)^2 \rfloor} (np_n)^2 \mathbb{P}(b_{n,1} = \lfloor x(np_n)^2 \rfloor | \zeta_n = k_n) dx. \end{aligned}$$

We show the pointwise convergence of the densities

$$\lim_{n \rightarrow \infty} (np_n)^2 \mathbb{P}(b_{n,1} = \lfloor x(np_n)^2 \rfloor | \zeta_n = k_n) = \frac{c}{\sqrt{2\pi x}} \exp\left(-\frac{c^2 x}{2}\right),$$

then Scheffé’s lemma implies that this convergence also holds in L^1 , which allows us to pass to the limit in the above integral:

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{-\lambda(np_n)^{-2} b_{n,1}} | \zeta_n = k_n] = \int_0^{\infty} \exp(-\lambda x) \frac{c}{\sqrt{2\pi x}} \exp\left(-\frac{c^2 x}{2}\right) dx = \mathbb{E}[e^{-\lambda c^{-2} Z^2}].$$

Recall from (9) the distribution of k_n i.i.d. Borel(1) random variables conditioned to have sum n : for any integers $n_1, \dots, n_{k_n} \geq 1$ such that $n_1 + \dots + n_{k_n} = n$,

$$\mathbb{P}(\beta_{n,1} = n_1, \dots, \beta_{n,k_n} = n_{k_n}) = \frac{(n - k_n)!}{k_n n^{n - k_n - 1}} \prod_{j=1}^{k_n} \frac{n_j^{n_j - 1}}{n_j!}.$$

In particular, the $\beta_{n,j}$'s are identically distributed and for any $m_n \in \{1, \dots, n - k_n + 1\}$, summing over all the $n_2, \dots, n_{k_n} \geq 1$ such that $n_2 + \dots + n_{k_n} = n - m_n$,

$$P(\beta_{n,1} = m_n) = \frac{(n - k_n)!}{k_n n^{n-k_n-1}} \frac{m_n^{m_n-1}}{m_n!} \frac{(k_n - 1)(n - m_n)^{n-m_n-k_n}}{(n - m_n - k_n + 1)!}.$$

Recall that on the event $\{\zeta_n = k_n\}$, $b_{n,1}$ is distributed as one of the variables $\beta_{n,1}, \dots, \beta_{n,k_n}$ chosen proportionally to its value minus 1. Therefore, for any $m_n \in \{1, \dots, n - k_n + 1\}$, we have

$$\begin{aligned} P(b_{n,1} = m_n | \zeta_n = k_n) &= \sum_{j=1}^{k_n} P(b_{n,1} = \beta_{n,j} | \beta_{n,j} = m_n) P(\beta_{n,j} = m_n) \\ &= k_n \frac{m_n - 1}{n - k_n} P(\beta_{n,1} = m_n) \\ &= (m_n - 1)(k_n - 1) \frac{(n - k_n - 1)!}{n^{n-k_n-1}} \frac{m_n^{m_n-1}}{m_n!} \frac{(n - m_n)^{n-m_n-k_n}}{(n - m_n - k_n + 1)!}. \end{aligned}$$

Suppose that $m_n, k_n \rightarrow \infty$ as $n \rightarrow \infty$ with $m_n, k_n = o(n)$, then Stirling's formula yields

$$\begin{aligned} P(b_{n,1} = m_n | \zeta_n = k_n) &= \frac{1}{\sqrt{2\pi}} \frac{k_n}{n\sqrt{m_n}} \exp\left(-\frac{k_n^2 m_n}{2n^2} + O\left(\frac{k_n^3 m_n}{n^3}\right) + O\left(\frac{(k_n m_n)^2}{n^3}\right)\right) (1 + o(1)). \end{aligned}$$

For any $x, c > 0$, taking $m_n = \lfloor x(np_n)^2 \rfloor$ and $k_n \sim cp_n^{-1}$, we obtain

$$\begin{aligned} (np_n)^2 P(b_n = \lfloor x(np_n)^2 \rfloor | \zeta_n = k_n) &= \frac{c}{\sqrt{2\pi x}} \exp\left(-\frac{c^2 x}{2} + O\left(\frac{1}{np_n}\right) + O(np_n^2)\right) (1 + o(1)) \\ &= \frac{c}{\sqrt{2\pi x}} \exp\left(-\frac{c^2 x}{2}\right) (1 + o(1)), \end{aligned}$$

and the proof is now complete. □

More generally, for any integer $j \geq 1$, denote by $\zeta_{n,j}$ the time of the j th fire, so that $b_{n,j}$ denotes the size of the subtree burnt at time $\zeta_{n,j}$.

Proposition 6. *We have for any $j \geq 1$,*

$$\lim_{n \rightarrow \infty} (np_n)^{-2} (b_{n,1}, \dots, b_{n,j}) = (\gamma_1^{-2} Z_1^2, \dots, \gamma_j^{-2} Z_j^2) \text{ in distribution.}$$

Proof. We prove the claim for $j = 2$ for simplicity of notation, the general case follows by induction in the same manner. Notice first that the times at which the first j fires appear jointly converge:

$$\lim_{n \rightarrow \infty} p_n(\zeta_{n,1}, \dots, \zeta_{n,j}) = (\gamma_1, \dots, \gamma_j) \text{ in distribution.} \tag{10}$$

Indeed, conditionally given the size of the first burnt subtree $b_{n,1} = m$ and the number of edges previously fireproof $\zeta_{n,1} - 1 = k - 1$, it remains a forest containing $(n - 1) - (k - 1) - (m - 1) = n - m - k + 1$ edges and the time $\zeta_{n,2} - \zeta_{n,1}$ we wait for the second fire after the first one is again a truncated geometric random variable which takes value

- ∞ with probability $(1 - p_n)^{n-m-k+1}$, and
- ℓ with probability $p_n(1 - p_n)^{\ell-1}$, for any $\ell \in \{1, \dots, n - m - k + 1\}$.

Since $b_{n,1} + \zeta_{n,1} = o(n)$ in probability, we see that $p_n(\zeta_{n,2} - \zeta_{n,1})$, conditionally given $b_{n,1}$ and $\zeta_{n,1}$, converges in distribution to an exponential random variable with parameter 1. This yields (10) in the case $j = 2$.

The same idea gives the claim of Proposition 6. The remaining forest after the first fire is, conditionally given $b_{n,1}$ and $\zeta_{n,1}$, uniformly distributed amongst the forests with $\zeta_{n,1} - 1$ trees and $n - b_{n,1}$ vertices. Therefore, conditionally given $b_{n,1}$ and $\zeta_{n,2}$, $b_{n,2}$ is distributed as the size of a tree chosen at random with probability proportional to its number of edges in a forest consisting of $\zeta_{n,2} - 1$ trees with total size $n - b_{n,1} \sim n$. Then the proof of Lemma 9 shows that such a random variable, rescaled by a factor $(np_n)^{-2}$, converges in distribution to $\gamma_2^{-2}Z_2^2$. This yields

$$\lim_{n \rightarrow \infty} (np_n)^{-2}(b_{n,1}, b_{n,2}) = (\gamma_1^{-2}Z_1^2, \gamma_2^{-2}Z_2^2) \quad \text{in distribution,}$$

and the proof is complete after an induction on j . □

5.2. Asymptotic size of all burnt subtrees

To strengthen the convergence from finite dimensional vectors to the ℓ^1 convergence, we need to bound the remainders. This is done in the following lemma, the last ingredient for the proof of Theorem 2.

Lemma 10. *For any $\varepsilon > 0$, we have*

$$\lim_{j_0 \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left((np_n)^{-2} \sum_{j=j_0}^{\infty} b_{n,j} > \varepsilon \right) = 0.$$

In order to prove this result, we consider a slightly different sequence of random subtrees of \mathbf{t}_n , which can be coupled with the sequence of burnt subtrees and for which the study is easier. Precisely, consider the following random dynamics on \mathbf{t}_n : we remove successively the edges in a random uniform order and at each step, we mark one subtree at random proportionally to its number of edges. We stress that in this procedure, the subtrees are not burnt, which implies that the edges of a marked subtree can be removed afterward and that a subtree of a marked one may be marked as well. For each $k = 1, \dots, n - 2$, we denote by $b'_{n,k}$ the size of the subtree which is marked when k edges have been removed.

Lemma 11. *There exists a numerical constant $C > 0$ such that for any $a > 0$, we have*

$$\limsup_{n \rightarrow \infty} n^{-2} p_n^{-1} \sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n-u_n \rfloor} \mathbb{E}[b'_{n,k}] \leq \frac{C}{a},$$

where $u_n = n \exp(-\sqrt{np_n})$ for every integer n .

The role of the sequences u_n and ap_n^{-1} shall appear in the proofs of Lemma 11 and Lemma 10; note that since $\lim_{n \rightarrow \infty} np_n = \infty$, we have

$$\lim_{n \rightarrow \infty} p_n u_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (np_n)^{-1} \ln \left(\frac{n - u_n}{u_n} \right) = 0. \tag{11}$$

The first convergence shows that the sum in Lemma 11 is not empty for n large enough.

Proof of Lemma 11. Fix $k \leq n - 2$ and let $(\beta_{n,1}, \dots, \beta_{n,k})$ be a k -tuple formed by i.i.d. Borel(1) random variables conditioned to have sum n . As we have seen, $b'_{n,k}$ can be viewed as one the $\beta_{n,i}$'s picked at random with probability proportional to its value minus one and hence,

$$\mathbb{E}[b'_{n,k}] = \mathbb{E} \left[\sum_{i=1}^k \beta_{n,i} \frac{\beta_{n,i} - 1}{n - k} \right] = \frac{n}{n - k} \mathbb{E} \left[\sum_{i=1}^k (\beta_{n,i} - 1) \frac{\beta_{n,i}}{n} \right].$$

Bertoin [4], Section 3.1, provides an upper bound for the expectation on the right-hand side. Precisely, Proposition 1 in [4], together with Lemma 5 and equation (2) there, shows that there exists a numerical constant $K > 0$ such that for every integers $1 \leq k \leq n$, we have

$$\mathbb{E} \left[\sum_{i=1}^k (\beta_{n,i} - 1) \frac{\beta_{n,i}}{n} \right] \leq K \left(\frac{n}{k} \right)^2.$$

Hence for every n ,

$$\sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n-u_n \rfloor} \mathbb{E}[b'_{n,k}] \leq Kn^3 \sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n-u_n \rfloor} \frac{1}{k^2(n-k)}.$$

Comparing sums and integrals, we have on the one hand,

$$\sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor 3n/4 \rfloor} \frac{1}{k^2(n-k)} \leq \frac{4}{n} \sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor 3n/4 \rfloor} \frac{1}{k^2} = \frac{4}{a} n^{-1} p_n (1 + o(1)),$$

and on the other hand,

$$\sum_{k=\lceil 3n/4 \rceil}^{\lfloor n-u_n \rfloor} \frac{1}{k^2(n-k)} \leq n^{-2} \left[\ln(x) - \ln(n-x) - \frac{n}{x} \right]_{3n/4}^{n-u_n} = n^{-2} \ln \left(\frac{n-u_n}{u_n} \right) (1 + o(1)).$$

Summing the two terms and appealing (11), we obtain

$$\sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n-u_n \rfloor} \mathbb{E}[b'_{n,k}] \leq \frac{4K}{a} n^2 p_n (1 + o(1)),$$

and the claim follows. □

We have a natural coupling between burnt and marked subtrees, which enables us to deduce Lemma 10 from Lemma 11: for each $k = 1, \dots, n - 2$, we toss a coin which gives Head with probability p_n ; the first burnt subtree, say, $\mathbf{t}_{n,1}$, is distributed as the first marked subtree, say, $\mathbf{t}'_{n,k}$, for which the outcome is Head. Then, the second burnt subtree $\mathbf{t}_{n,2}$ is distributed as the next marked subtree for which the outcome is Head and which is not contained in $\mathbf{t}'_{n,k}$, and so on. In the next proof, we implicitly assume that the marked and burnt subtrees are indeed coupled.

Proof of Lemma 10. Fix $\varepsilon, \delta > 0$ and $a > \delta^{-1}$. Since $p_n \zeta_{n,j} \rightarrow \gamma_j$ in distribution as $n \rightarrow \infty$ for any $j \geq 1$ and $j^{-1} \gamma_j \rightarrow 1$ in probability as $j \rightarrow \infty$ from the law of large numbers, we may, and do, fix $j_0 \geq 1$ and further $n_0 \geq 1$ such that for any $n \geq n_0$, we have

$$\mathbb{P}(\zeta_{n,j_0} > ap_n^{-1}) \geq 1 - \delta.$$

For any $j \geq 1$, denote by $\theta_{n,j} - 1$ the number of edges that have been removed in the marking procedure when we mark the subtree corresponding to the burnt subtree $\mathbf{b}_{n,j}$. We have

$$\sum_{j \geq j_0} \mathbf{b}_{n,j} \leq \sum_{k=1}^{n-2} \mathbf{b}'_{n,k} \mathbf{1}_{\{\eta_k=1\}} \mathbf{1}_{\{k \geq \theta_{n,j_0}\}},$$

where $\eta_k = 1$ if and only if the outcome of the coin which is tossed at the k th step is Head. Further, since $\zeta_{n,1} = \theta_{n,1}$ and $\zeta_{n,j} \leq \theta_{n,j}$ for every $j \geq 2$, we see that

$$\mathbb{P}\left((np_n)^{-2} \sum_{j=j_0}^{\infty} \mathbf{b}_{n,j} > \varepsilon \mid \zeta_{n,j_0} > ap_n^{-1}\right) \leq \mathbb{P}\left((np_n)^{-2} \sum_{k=\lfloor ap_n^{-1} \rfloor}^{n-2} \mathbf{b}'_{n,k} \mathbf{1}_{\{\eta_k=1\}} > \varepsilon\right).$$

Recall that $\lim_{n \rightarrow \infty} p_n u_n = 0$, which implies that the probability that no tree is marked after the $\lfloor n - u_n \rfloor$ th step is $(1 - p_n)^{\lfloor u_n \rfloor - 2} \geq 1 - \delta$ for any n large enough. Finally, from Lemma 11,

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left((np_n)^{-2} \sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n - u_n \rfloor} \mathbf{b}'_{n,k} \mathbf{1}_{\{\eta_k=1\}} > \varepsilon\right) \leq \limsup_{n \rightarrow \infty} \varepsilon^{-1} (np_n)^{-2} \sum_{k=\lfloor ap_n^{-1} \rfloor}^{\lfloor n - u_n \rfloor} \mathbb{E}[\mathbf{b}'_{n,k}] p_n \leq \varepsilon^{-1} \frac{C}{a}.$$

We conclude that

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left((np_n)^{-2} \sum_{j=j_0}^{\infty} \mathbf{b}_{n,j} > \varepsilon\right) \leq \varepsilon^{-1} C \delta + 2\delta,$$

and the claim follows since δ is arbitrary. □

Using the same reasoning as in the proof of Theorem 1, Theorem 4 follows readily from Proposition 6 and Lemma 10.

Proof of Theorem 4. For every $n, j \geq 1$, we write $((np_n)^{-2} \mathbf{b}_{n,k})_{k \geq 1} = S_n(j) + R_n(j)$ with

$$S_n(j) = (np_n)^{-2} (\mathbf{b}_{n,1}, \mathbf{b}_{n,2}, \dots, \mathbf{b}_{n,j}, 0, 0, \dots)$$

and

$$R_n(j) = (np_n)^{-2} (0, \dots, 0, \mathbf{b}_{n,j+1}, \mathbf{b}_{n,j+2}, \dots);$$

and similarly, $(\gamma_i^{-2} Z_i^2)_{i \geq 1} = S(j) + R(j)$, with

$$S(j) = (\gamma_1^{-2} Z_1^2, \gamma_2^{-2} Z_2^2, \dots, \gamma_j^{-2} Z_j^2, 0, 0, \dots)$$

and

$$R(j) = (0, \dots, 0, \gamma_{j+1}^{-2} Z_{j+1}^2, \gamma_{j+2}^{-2} Z_{j+2}^2, \dots).$$

From Proposition 6, for any $j \geq 1$, $\lim_{n \rightarrow \infty} S_n(j) = S(j)$ in distribution. Further, for any $\varepsilon > 0$, since the sequence $(\gamma_i^{-2} Z_i^2)_{i \geq 1}$ is summable, and thanks to Lemma 10, there exists $j_0 \geq 1$ and then $n_0 \geq 1$ such that

$$\mathbb{P}(\|R(j_0)\| > \varepsilon) < \varepsilon \quad \text{and} \quad \text{for every } n \geq n_0, \quad \mathbb{P}(\|R_n(j_0)\| > \varepsilon) < \varepsilon,$$

which completes the proof. □

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