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A few aspects of large random maps

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Contents

Introduction

The aim of this document is to present my research work carried out since my PhD, compare the papers to each others, and replace them in the scientific literature. It is not meant to be self-contained at all! We shall only present some results and sketch their proofs by giving the main ideas, when not omitting them entirely. The reader can find the detailed arguments in the articles, which are listed below. In a few words, my entire work consists in studying random discrete structures and their asymptotic behaviour as their 'size' tends to infinity. During my PhD. I considered random trees and sort of fragmentation processes on them [1] [2] as well as non-crossing configurations of the disk (non-crossing partitions and trees) [3] [4]. Although there are still open questions in these fields which I would like to study, starting from early 2016, I moved on to considering random maps, which will be the topic of this document.

1 List of publications

Below is a complete list of my (pre-)publications, starting from the most recent one but numbered in the order of their completion (say their first appearance on arXiv). They are duplicated in the general bibliography which can be found in the end of this document, labelled by authors' initials+date; this is how they will be cited after this introduction. For example [14] here will later be cited as [CKM22].

Publications after the PhD.

[15] Scaling limits of random looptrees and bipartite plane maps with prescribed large faces
C. Marzouk. *Preprint available at* arXiv:2202.08666, 2022.

[14] The mesoscopic geometry of sparse random maps
N. Curien, I. Kortchemski, & C. Marzouk. *J. Éc. polytech. Math.*, 9:1305–1345, 2022.

[13] Large deviation Local Limit Theorems and limits of biconditioned Trees and Maps
I. Kortchemski & C. Marzouk. *To appear in* Ann. Appl. Probab. *Preprint available at* arXiv:2101.01682, 2021.

[12] Infinite stable Boltzmann planar maps are subdiffusive
N. Curien & C. Marzouk. *Prob. Math. Phys.*, 2(1):1–26, 2021.

[11] On scaling limits of random trees and maps with a prescribed degree sequence
C. Marzouk. *Annales Henri Lebesgue*, 5:317–386, 2022.

[10] Markovian explorations of random planar maps are roundish
N. Curien & C. Marzouk. *Bull. Soc. Math. France*, 148(4):709–732, 2020.

[9] On scaling limits of planar maps with stable face degrees
C. Marzouk. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15:1089–1122, 2018.

[8] Scaling limits of discrete snakes with stable branching
C. Marzouk. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(1):502–523, 2020.

[7] How fast planar maps get swallowed by a peeling process
N. Curien & C. Marzouk. *Electron. Commun. Probab.*, 23(18):1–11, 2018.

[6] Infinite random planar maps related to Cauchy processes
T. Budd, N. Curien, & C. Marzouk. *J. Éc. polytech. Math.*, 5:749–791, 2018.

[5] Scaling limits of random bipartite planar maps with a prescribed degree sequence
C. Marzouk. *Random Struct. Alg.*, 53(3):448–503, 2018.

Earlier publications, from the Ph.D.

[4] Triangulating stable laminations
I. Kortchemski & C. Marzouk. *Electron. J. Probab.*, 21(11):1–31, 2016.

[3] Simply generated non-crossing partitions
I. Kortchemski & C. Marzouk. *Combin. Probab. Comput.*, 26(4):560–592, 2017.

[2] Fires on large recursive trees
C. Marzouk. *Stochastic Process. Appl.*, 126(1):265–289, 2016.

[1] On the sizes of burnt and fireproof components for fires on a large Cayley tree
C. Marzouk. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(1):355–375, 2016.

The rest of this introduction is dedicated to a brief presentation of the organisation of the next sections. They are split into two chapters, the first one focusing on *scaling limits* of random maps and the second one on their *local limits*.

2 Scaling limits of random maps

This corresponds to the papers [5] [8] [9] [11] [13] [15].

After about a decade of work, starting with the pioneer article by Chassaing & Schaeffer [CS04], Le Gall [LG13] and Miermont [Mie13] independently closed the proof of the convergence in distribution of quadrangulations of the sphere with n faces sampled uniformly at random, when rescaled by $n^{1/4}$, to a continuum random surface called the *Brownian map*, which is nowadays more often referred to as *Brownian sphere*. Following this result, and indeed relying on it, the Brownian sphere has been shown to appear as the limit of many different random maps [LG13, BLG13, BJM14, Abr16, ABA17, ABA21] and the first chapter is dedicated to showing the convergence of more models to this object, as well as to attempt to describe other limits.

A model with prescribed face degrees [5] [11] [15]

One natural way to extend the model of quadrangulations is to fix for every integer $n \geq 1$ the degrees of the n faces that we want and sample a plane map uniformly at random with this constraint. This model, as often restricted to the bipartite setting, is introduced in [5] and is inspired by similar plane trees sampled uniformly at random with given offspring numbers [BM14]. In [5] the Brownian sphere is obtained as the scaling limit under the same assumptions on the degrees used in [BM14] to prove the convergence to Aldous' Brownian CRT. The proof actually relies on this work by relating such trees with random labels to our maps via the combination of the bijections from [BDFG04] and [JS15]. This result is completely generalised in [11] where, first, the growth rate of such maps is determined in full generality and second, a simple necessary and sufficient condition on the degrees is given for these maps to converge to the Brownian sphere. The case where this assumption fails is considered in [15] by rephrasing the previous bijection in terms of *looptrees à la* Curien & Kortchemski [CK14], which simplifies the study.

Stable maps and snakes [8] [9]

Boltzmann random maps are models in which the face degrees are random, and in a sense i.i.d. introduced by Marckert & Miermont [MM07] partly to probe the universality of the Brownian sphere. In addition they are related to statistical physics models on maps. By forcing large faces to appear, they provide other universality classes known as *stable maps* [LGM11], although the convergence is only known along subsequences for the moment. In [9] we extend these results to the full domain of attraction of stable laws; one motivation was to simplify some of the arguments from [5] and adapt them to models with large faces. A key relation with the previous model is that, conditional on the face degrees, the maps have the uniform distribution and [5] already provided the convergence of maps in the finite variance regime. The subsequent papers [11] and [15] then allow to fully recover [9]. In another direction, maps are related to labelled trees and in the Brownian regime, the limit is the celebrated Brownian snake which can be seen as a branching Brownian motion with genealogical structure given by the Brownian tree. In the stable regime however the limit is not a branching Brownian motion with genealogical structure given the stable tree. This object is obtained in [9] as the limit of discrete models not related to maps and this paper is briefly presented here just to point out the differences with the other model. Interestingly the technical challenges are quite different.

Biconditioned maps [13]

In the aforementioned scaling limits of Boltzmann maps, one conditions such maps to have either n vertices, or n edges, or n faces and let $n \to \infty$. The three choices all lead to the same results, up to a multiplicative constant, essentially because in these large maps, they are typically linearly related to each other. Recently with Igor Kortchemski we asked about the behaviour of Boltzmann maps conditioned by these three quantities at the same time when they tend to infinity outside these linear relations. By Euler's formula it actually leaves two parameters, hence the name. This model does not seem to have been studied previously, only Fusy & Guitter [FG14] predicted the growth rate of such maps for the uniform distribution, which we confirm and furthermore prove that the limit is always the Brownian sphere. We rely for this on [11] and from a technical point of view, this work [13] has little to do with maps and is more about controlling Łukasiewicz paths conditioned on having a given length and a given total number of downward increments. As a key technical input of independent interest, we obtain local limit estimates for random walks in new regimes.

3 Local limits of random maps

This corresponds to the papers [6] [7] [10] [12] [14].

A very different point of view on maps is that of *local limits* where one lets the size of the map tend to infinity without scaling distances, so the limit is now a discrete infinite map. Uniformly chosen triangulations and quadrangulations were first proved to converge in this sense by Angel & Schramm [AS03] and Krikun [Kri05] respectively and the more general stable Boltzmann maps have been considered by Björnberg & Stefánsson [BS14] and Stephenson [Ste18] who provide the existence of infinite stable maps, whose large scale behaviour is dictated by a parameter $a \in (3/2, 5/2]$, where the case $a = 5/2$ contains the triangulations and quadrangulations, and more generally all 'finite variance' regimes. We are then interested in the behaviour of these maps: their geometry, percolation, random walk, etc.

Generalities on the peeling of stable maps [7] [10]

A key tool to study these infinite stable maps is an exploration procedure that somehow reveals the map face by face known as *peeling process*, first constructed by Angel [Ang03] on triangulations and then

extended by Budd [Bud16] to deal with unbounded face degrees. The strength of these processes is that one has a lot of freedom in the way to explore the map and which face to reveal at each step, but on the other hand many properties do not depend on this choice and Budd [Bud16] shows that the perimeter and volume of the exploration always has the same law. In the works [7] and [10] in collaboration with Nicolas Curien, we aimed at considering more geometric properties shared by all these explorations. In the first short paper we give a universal upper bound for the time the starting point stays on the boundary of the exploration and in the second one we prove that a peeling exploration always grows roughly like a (completed) metric ball.

The dual stable maps and the intermediate regime [6]

Timothy Budd & Nicolas Curien [BC17] studied specifically the behaviour of the dual maps of these infinite stable maps and showed, in accordance with the physics literature, that the so-called *dense regime* $a < 2$ and *dilute regime* $a > 2$ are very different. Together we then studied the boundary case $a = 2$ and proved that its behaviour is intermediate between the two others. For example while the volume growth of the (completed) balls is polynomial when $a > 2$ and exponential when $a < 2$, we prove that it is exponential in the square-root of the radius when $a = 2$. We also study percolation and first-passage percolation on this model. This is all based on careful uses of the peeling exploration. It raised many technical challenges that one can catch a glimpse of by seeing that the perimeter of the exploration is related to a random walk attracted to a Cauchy process, which is often left aside in the literature.

The random walk is subdiffusive [12]

Understanding the behaviour of random processes on maps often sheds some lights on its geometric properties. After a pioneer work by Benjamini & Schramm [BS01] conjecturing the simple random walk on triangulations to be recurrent, this was finally proved by Gurel-Gurevich & Nachmias [GGN13] whose criterion also applies to quadrangulations and actually Boltzmann maps [BS14, Ste18]. In another direction, Benjamini & Curien [BC13] exhibited a subdiffusive behaviour for the walk on quadrangulations, showing that in n steps, it only displaces by at most $n^{1/3}$ so in a sense these graphs design many dead ends which trap the walk, as opposed to regular lattices (where the walk moves at speed $n^{1/2}$). Using very different ideas, although still relying on peeling explorations, we obtained in [12] in collaboration with Nicolas Curien this $1/3$ upper exponent for all values of $a \in (3/2, 5/2]$.

Biconditioned maps again, yet a littler closer [14]

In this paper in collaboration with Nicolas Curien & Igor Kortchemski, we consider uniform (now non necessarily bipartite) random maps with a fixed number of vertices, edges, and faces, in a 'sparse' regime where the number of faces is much smaller than the number of edges, so their degrees typically tend to infinity. As opposed to [13], we aim here at understanding their asymptotic geometry in an intermediate scale where the faces remain macroscopic and we bring back together scaling and local limits. We use a completely different method from all the aforementioned papers, relying on a so-called core–kernel decomposition, to obtain a semi-continuous limit built from a discrete infinite map in which each edge is replaced by a random continuum tree structure.

Chapter 1

Scaling limits: Universality of the Brownian sphere and beyond

This chapter presents the papers [Mar18b, Mar20, Mar18a, Mar22a, KM21, Mar22b] pertaining to the theory of scaling limits of random maps. It is organised as follows.

Section 1.1. We recall the basic definitions and the notation that we will use throughout this document as well as the Gromov–Hausdorff–Prokhorov topology which is used in this first chapter. We then recall the road map to the convergence of random quadrangulations, starting with the Schaeffer representation by a labelled tree, the convergence of this random labelled tree to the Brownian snake [CS04], and eventually the convergence of quadrangulations to the Brownian sphere [LG13, Mie13], with some emphasis on the *rerooting trick* which will play a major role since it makes extensions of this result to other models of random maps much simpler.

Section 1.2. We describe the papers [Mar18b, Mar22a, Mar22b] on the model of uniform random (bipartite) maps with prescribed face degrees and the convergence of such maps following the recipe described in Section 1.1. We first present a generalisation of the Schaeffer bijection by reformulating those from [BDFG04] and [JS15] in terms of labelled looptrees. The main point is that the geometry of a looptree is explicitly coded by a Łukasiewicz path, which is much simpler to study than the height or contour process of a tree. This lifts several technical difficulties and allows us to derive tightness of random labelled looptrees and maps in full generality by relying on the Łukasiewicz paths. Then under natural assumptions under which these paths are known to converge to excursions of exchangeable increment processes, we prove the convergence of the labels by treating separately the increments on large and small loops/increments of the Łukasiewicz path. The consequences for maps then follow as described in Section 1.1.

Section 1.3. We recall the model of random (bipartite) Boltzmann maps, in which the face degrees are random and roughly speaking i.i.d. When their common distribution belongs to the domain of attraction of a stable law, they converge, only along subsequences outside the Gaussian regime, to stable maps [Mar18a], which slightly extends the pioneer work of Le Gall & Miermont [LGM11]. Actually, by seeing Boltzmann maps as mixtures of maps sampled with prescribed face degrees, in which these degrees are themselves random, one can recover these results from [Mar22a, Mar22b]. This strategy also allows us to consider maps conditioned by both their number of vertices and of edges (and thus of faces as well). Thanks to [Mar22a], convergence to the Brownian sphere boils down to studying the degree distribution of the faces, which correspond to the increments of the associated Łukasiewicz path, which is a downward skip free excursion conditioned by its total length and its number of negative steps. In particular we want to prove the convergence of such a path to the Brownian excursion, with

Igor Kortchemski [KM21] we provide sufficient conditions in terms of both the law of the walk and the proportion of negative increments to ensure this convergence. An important technical input is a suitable local estimate which compares in a strong sense the mass function of a random walk at time n with the Gaussian density, which we prove in new regimes.

1.1 The fundamental case of quadrangulations

The aim of this section is to provide some necessary background on the theory of scaling limits of random maps; our personal contribution to this topic is presented in the next two sections. We shall however stay brief. We learnt this during the lectures of Grégory Miermont in Saint-Flour, so we naturally refer to his lecture notes [Mie14] for details.

1.1.1 Basic definitions, notations, and conventions

Maps as embedded graphs

As a first definition, a plane map is the embedding on the sphere of a finite connected multigraph (allowing loops and multiple edges), without edge-crossing, and viewed up to orientation-preserving homeomorphisms of the surface, see Figure 1.1 for an example. Let us emphasise the difference between a planar map and a planar graph: the latter is an abstract graph that *can* be embedded in the plane, whereas the former is *one choice of such embeddings*, up to continuous deformations. From a combinatorial point of view, the embedding can also be described by a local ordering of all the half-edges incident to a vertex.

Due to the embedding, in addition to the vertices and edges of the associated graph, it also makes sense to define the faces, which are the connected components of the complement of the map on the sphere. For a map M, we shall denote by $V(M)$, $E(M)$, and $F(M)$ the set of its vertices, edges, and faces. The *degree* of a vertex v and of a face f, denoted by $\deg(v)$ and $\deg(f)$ respectively, is the number of incident half-edges or corners, or in other words the number of edges counted with multiplicity. Let us mention here the celebrated Euler formula: Any map M on the sphere satisfies

$$\#V(M) + \#F(M) = \#E(M) + 2.$$

Note in particular that when all faces have the same degree, say $p \geq 3$ in which case the map is called a 'p-angulation', if the map has n such faces, it necessarily has $pn/2$ edges and thus $(p-2)n/2 + 2$ vertices. For example, for $p = 4$, *quadrangulations* with n faces have $2n$ edges and $n + 2$ vertices.

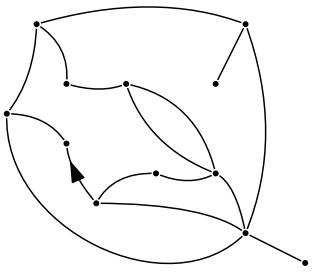


Figure 1.1 – A (rooted plane) map with six faces with degree 8, 8, 6, 4, 4, 2; the outer face plays no role and simply comes from the representation on the plane rather than on the sphere.

All our maps will be *rooted*, by which we mean that we distinguish an oriented edge, hereafter called *root edge* and denoted by $\vec{e}$. Another equivalent convention is to distinguish a half-edge or a corner instead. The reason is to break the otherwise numerous symmetries (automorphisms) in the maps, as well as to anchor the maps at a starting point. We shall call the vertex ρ at the origin of the root edge the *root vertex*, and the face f_{r} lying to its right the *root face*. Below, by the sole term 'map' we

shall always refer to rooted plane maps. Only in Section 2.3.3, shall we consider maps on other surfaces, namely tori with high genus.

We shall often consider *pointed* maps, in which a vertex is distinguished, usually denoted by $v_\star$. It will be convenient to view the root face as a *boundary*, which is not necessarily simple; its degree is then called the *perimeter* of the map. A map without a boundary can be seen as a map with a boundary of perimeter 2 by simply adding a parallel edge to the right of the root edge, and conversely, in a map with a boundary of perimeter 2, gluing tother these two edges yields a map without boundary.

Finally, if planar maps are simpler to study than planar graphs because they are more rigid, an even more rigid class of maps is that of *bipartite* maps which are maps in which the vertices can be coloured using two colours in such a way that an edge always links a vertex of one colour with another one of the other colour. In the case of plane maps, this is equivalent to requiring all the faces to have even degree. We shall work under this technical restriction in this entire document, until Section 2.3.

Other definitions

Equivalently to the preceding definition, a plane map is a cellular decomposition of the sphere. Namely suppose we are given a list of polygons with an even number of sides in total, then we produce a surface by matching these sides by pairs, and we restrict to those matchings that create a topological sphere, as represented in Figure 1.2. The pairs of sides matched become the edges of the map, the polygons become the faces, their number of sides is the degree of the corresponding face.

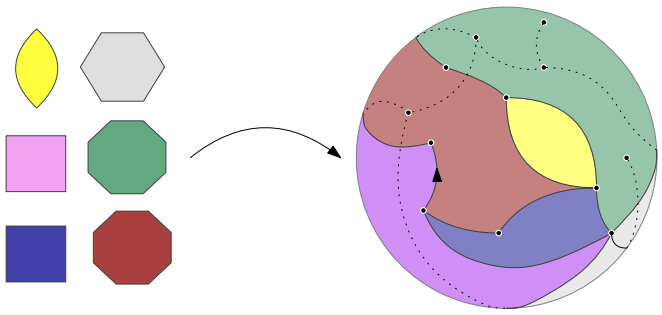


Figure 1.2 – The map from Figure 1.1 viewed as a gluing of polygons.

This point of view on maps will be used to get an intuition on the models as well as to understand one of the motivations of studying random maps, namely to produce models of discrete random surfaces with a given topology. The first definition as embedded graphs will be used in the statements and proofs. Let us very briefly mention the existence of a third definition of maps as pairs of permutations. Indeed, label each side of each edge (or equivalently of each side in the list of polygons before the gluing) from 1 to say $2n$, then the cyclic ordering of these labels when turning around each face in clockwise order forms the cycles of a first permutation, while the pairs of labels of each edge (equivalently the matching of the sides of the polygons) form an involution with no fixed point. This will not at all be used in this document but it provides an algebraic approach to maps which has been very fruitful.

1.1.2 The Gromov–Hausdorff–Prokhorov topology

Our aim is to study the limits of large maps. The topology we use in this entire first part is the *Gromov–Hausdorff–Prokhorov topology*, abbreviated from now on GHP topology. It does not capture all the information on the map, especially the embedding on the sphere. Instead, given a finite map M, we only retain the triple given by its set of vertices $V(M)$, the graph distance d_M on that set, i.e. for each pair of vertices, the number of edges on the shortest path between them, and finally the uniform probability measure on $V(M)$. We still denote by M this space, and more generally, for $c > 0$, we will denote by cM the same space in which the graph distances are multiplied by c.

Therefore, scaling limits of maps in this sense can be seen as a first step towards two directions that we will not even approach here but are certainly important questions for the next years.

(i) Scaling limits of planar *graphs* in the GHP topology. This is more complicated due to the lack of rigidity and good representations compared to maps. In this direction, let us mention the very recent independent works by Stufler [Stu22a, Stu22b] and by Albenque, Fusy, & Lehéricy [AFL22] who proved the convergence to the Brownian sphere of random *trivalent* (or cubic) graphs. These results rely on the convergence of maps via Whitney's theorem which shows that three-connected planar graphs only have two planar embeddings (one being the mirror image of the other), so they can really be seen as (three-connected) maps. Thus trivalent such graphs relate by duality to simple triangulations, studied in [ABA17]. Non three-connected graphs are then studied by surgery, and compared to their three-connected core.

(ii) Scaling limit of maps canonically embedded on the sphere. Since the GHP topology forgets the embedding, then even the fact that the Brownian sphere indeed has the topology of the sphere does not follow from the convergence but from extra arguments [LGP08, Mie08b] and it does not come clearly with a canonical embedding on the sphere. The conjectural limits of embedded maps are the γ-Liouville Quantum Gravity surfaces of Duplantier & Sheffield [DS11] with $\gamma \in (0, 2)$. Actually, LQGs are random measures on the sphere, but Miller & Sheffield argued directly in the continuum world in a series of papers [MS20, MS21b, MS21c] that for $\gamma = \sqrt{8/3}$, one can define a metric structure which is that of the Brownian sphere, and conversely that the metric structure of the Brownian sphere determines a conformal structure and so an embedding on the sphere, and thus induces a random measure which is the $\sqrt{8/3}$-LQG. Later Holden & Sun [HS19] constructed an embedding of random triangulations with a large simple boundary which they proved to converge to $\sqrt{8/3}$-LQG; they use for this the GHP convergence which was previously obtained in [AHS20].

The abstract setting

Let us introduce very briefly the topological setting and refer to [Mie09, Section 6] for details. In words, two compact metric spaces equipped with a Borel probability measure, say (X, d_X, p_X) and (Y, d_Y, p_Y), are close to each other in the GHP topology if one can find a subset of each which carries most of the mass and which are close to be isometric. More formally, let the GHP distance between X and Y be the infimum of all the values $\varepsilon > 0$ such that there exists a metric space (Z, d_Z) and two isometric embeddings $\phi : X \longrightarrow Z$ and $\phi' : Y \longrightarrow Z$ such that both the Hausdorff distance between the compact subsets $\phi(X)$ and $\phi'(Y)$ of Z and the Prokhorov distance between the push-forward laws $\phi_* p_X$ and $\phi'_* p_Y$ are less than ε.

A useful equivalent definition in practice is as follows. A *correspondence* between X and Y is a subset $R \subset X \times Y$ such that for every $x \in X$, there exists $y \in Y$ such that $(x, y) \in R$ and vice-versa. The *distortion* of R is defined as

$$\mathrm{dis}(R) = \sup \left\{ \left| d_X(x, x') - d_Y(y, y') \right| ; (x, y), (x', y') \in R \right\}.$$

Then the GHP distance between X and Y is the infimum of all the values $\varepsilon > 0$ such that there exists a coupling $p_{X,Y}$ between p_X and p_Y and a compact correspondence R between X and Y such that

$$p_{X,Y}(R) \geq 1 - \varepsilon \qquad \text{and} \qquad \mathrm{dis}(R) \leq 2\varepsilon.$$

The GHP distance is actually only a pseudo-distance, but after taking the quotient by measure-preserving isometries, one gets a genuine distance which is separable and complete.

Remark 1.1. Let us point out that many works only consider the Gromov–Hausdorff (GH) distance on compact metric spaces without probability measure. This topology is defined as above by simply

removing the condition on the Prokhorov distance or on the mass of the correspondence. We shall see that one has to a bit careful to take into account the measures, but it does not add any technicality.

A concrete setting with the example of trees: the distance function

The proofs of convergence of random maps often provide a concrete space in which they are embedded, namely the interval $[0,1]$. Let us illustrate this in the simpler context of plane trees, i.e. maps with a single face. Indeed it is well-known that a plane tree T^n with $n+1$ vertices can be coded by a Dyck path, i.e. a discrete path started at 0, with n increments equal to -1 and n increments equal to 1, and which always stays nonnegative. See Figure 1.4 top for an example. In words, one can associate with each positive increment the first negative subsequent increment at the same height, and merging these pairs together gives rise to the edges of the tree. The Dyck path is known as the *contour process* of the tree and is denoted by C^n; the name comes from the fact that C^n may be constructed from T^n by following its contour from left to right and reporting the distances to the root vertex.

The time component in C^n then corresponds to the sequence of successive corners of T^n, say $c_0, \dots, c_{2n}$, with $c_0 = c_{2n}$ and the distances between the vertices incident to these corners (which we still denote by c_i) in T^n are explicitly encoded by C^n. Indeed the distance between c_i and c_j is the sum of the distance from each to their last common ancestor, and this equals

$$d_{T^n}(i,j) = C^n_i + C^n_j - 2 \min_{i\wedge j \le k \le i\vee j} C^n_k.$$

Now given any continuous function $h : [0,1] \longrightarrow \mathbf{R}$, let us set similarly for every $s, t \in [0,1]$,

$$d_h(s,t) = h(s) + h(t) - 2 \min_{[s\wedge t, s\vee t]} h. \tag{1.1}$$

One can check that it is a pseudo-distance on $[0,1]$, we then define the quotient space

$$\mathcal{T}_h \;=\; [0,1] \big/ \{d_h = 0\}, \tag{1.2}$$

called a compact *real tree* in the literature, see [LGo5] and references therein. It has a natural probability measure given by the push-forward of the uniform Lebesgue measure by the canonical projection π_h, and it can be seen as rooted at $\pi_h(0)$. Note that if $h^n = C^n_{2n\cdot}$, then $\mathcal{T}_{h^n}$ is simply (isometric to) the tree T^n in which we replaced every edge by copies of the segment $[0,1]$, glued at their extremities according to the graph structure.

Provided that there exists a scaling factor $\alpha_n \longrightarrow 0$ such that

$$(\alpha_n C^n_{2nt}; t \in [0,1]) \underset{n\to\infty}{\longrightarrow} (h_t; t \in [0,1]) \quad \text{in the uniform topology,}$$

then $\alpha_n d_{T^n}((2n)^{-1}\cdot, (2n)^{-1}\cdot)$ converges to d_h for the uniform topology as well and this easily implies that

$$\alpha_n T^n \underset{n\to\infty}{\longrightarrow} \mathcal{T}_h \quad \text{in the GH topology,}$$

by considering the correspondence $R = \{(c_{\lfloor 2nt \rfloor}, \pi_h(t)); t \in [0,1]\} \in V(T^n) \times \mathcal{T}_h$.

Following Remark 1.1, this does not directly imply the convergence in the GHP topology because $(c_0, \dots c_{2n})$ lists the vertices of T^n *with redundancies*, namely each vertex appears as many times as its total degree (plus one for the root), so if U has the uniform distribution on $[0,1]$, then $c_{\lfloor 2nU \rfloor}$ is not a uniform random vertex a priori. This is easily solved in this simple case: note that each vertex of T^n different from the root corresponds to exactly two increments of C^n, namely the two that are matched to construct the edge linking it to its parent. Then just replace $\lfloor 2nt \rfloor$ in the correspondence R above by $\lceil 2nt \rceil$ whenever $2nt$ lies on a positive increment.

1.1.3 Quadrangulations as labelled trees

The CVS bijection

The enumeration of maps started with the seminal work of Tutte [Tut62b, Tut62a, Tut63] who provided explicit and quite simple formulae for different families of maps using combinatorial decompositions to derive functional equations on the generating series and then actually solving these equations. This approach is still used and Tutte's techniques have been generalised, but a bijective approach has also been developed, which consists in drawing an explicit correspondence between families of maps and simpler objects, easier to count.

In the case of quadrangulations, building on the work of Cori and Vauquelin [CV81], Schaeffer [Sch98] provided a 2-to-1 correspondence between pointed quadrangulations with n faces and *well labelled* trees with n edges, i.e. plane trees whose vertices carry integer labels which differ by $\{-1, 0, 1\}$ along the edges. The 2-to-1 comes from the orientation of the root edge of the quadrangulation which is lost. His construction in both directions is very explicit, let us only represent it on an example, in Figure 1.3; we shall present more precisely in Section 1.2.2 generalisations to all bipartite plane maps.

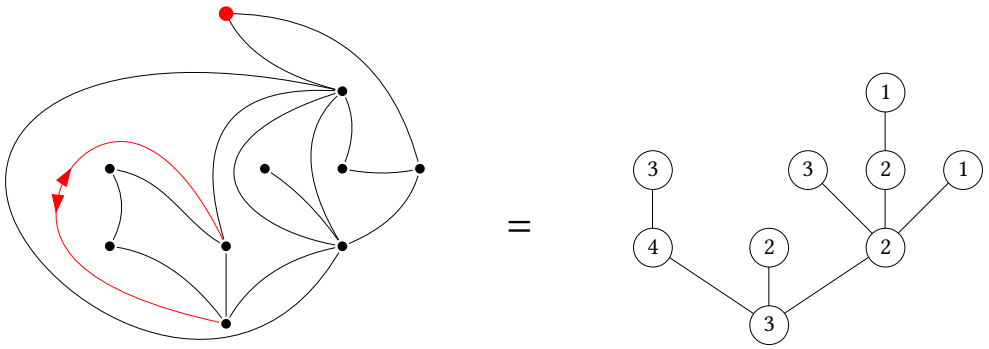


Figure 1.3 – Schaeffer's bijection on an example; on the left the quadrangulation is pointed at the red vertex on top, and rooted at the red edge in one of the two possible directions; on the right a well labelled tree whose labels have been shifted so the minimum is 1, in which case they represent the distances to the pointed vertex in the quadrangulation.

The first advantage of this construction is that it makes Tutte's enumeration formula crystal clear: recall that a quadrangulation with n faces has $n+2$ vertices, then there are

$$\frac{2 \cdot 3^n}{n+2} \cdot \mathrm{Cat}_n,$$

non-pointed quadrangulations of the sphere with n faces, where $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n'th Catalan number, which notably enumerates Dyck paths and trees (and thousands of other objects). Indeed, the factor 2 comes from the fact that we have a 2-to-1 correspondence, the 3^n from the three possible label increments along each edge of the tree, and the $n+2$ in the denominator, from the $n+2$ possible distinguished vertices in the pointed map.

Second, this correspondence opens the possibility to study quadrangulations sampled uniformly at random by considering the associated random labelled trees. Indeed, the key property of this correspondence is that the vertices of the tree correspond to those of quadrangulation different from the distinguished one $v_\star$, and moreover if one shifts all labels so the minimum is 1, then the label of every vertex equals its graph distance to $v_\star$ in the quadrangulation. Note that a uniformly sampled pointed quadrangulation has the same law as a uniformly sampled non-pointed quadrangulation in which one distinguish a vertex independently and uniformly at random among the $n+2$ possibilities. Then the random labelled tree directly gives access to the graph distances to a uniform point in the quadrangulation, which constitutes a first key step towards controlling all the pairwise distances.

Random labelled trees and the Brownian snake

Recall the coding of a plane tree T^n with n edges by a Dyck path C^n of length $2n$ mentioned above. If the tree is labelled, which we denote by a function $\ell^n : V(T^n) \longrightarrow \mathbf{Z}$, then let us label each positive increment of the path by the label increment of the corresponding edge of the tree and each negative increment by its opposite. Then the cumulative sum of the labels on the contour process define the *label process* and is denoted by L^n. We extend both C^n and L^n by linear interpolation. See Figure 1.4 for an example.

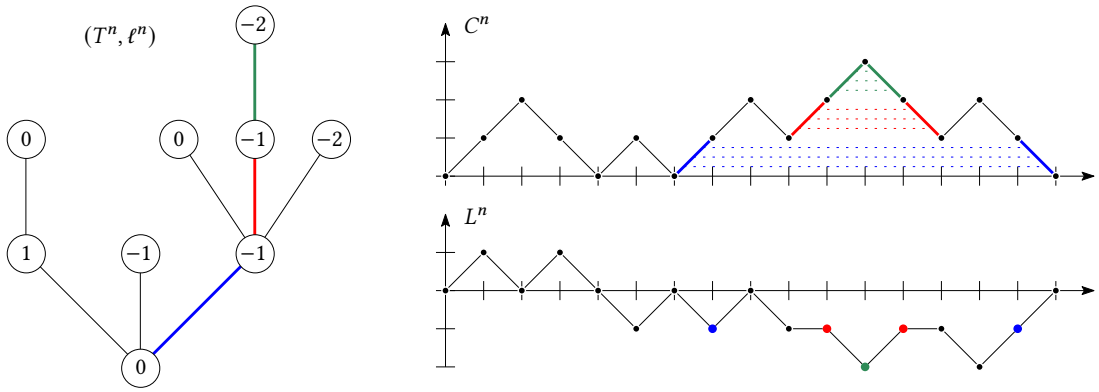


Figure 1.4 – Left: a labelled tree (T^n, ℓ^n). Right: its contour process C^n on top and its label process L^n below. The three coloured edges in T^n correspond to the coloured pairs of increments in C^n and the label of the vertices on top of them is the value of L^n at the instant of the top of the increments.

The random labelled tree (T^n, ℓ^n) associated with a uniform random quadrangulation is simply a uniform random plane tree with n edges equipped with a branching lazy random walk in the sense that the root of the tree has label 0 and the label increments are i.i.d. random variables which take value either -1, 0, or 1 with probability $1/3$ each. In this case the contour process C^n has the same law as an excursion of the simple random walk and it is classical that the convergence in distribution

$$\left(\frac{1}{\sqrt{2n}} C^n_{2nt}; t \in [0,1] \right) \xrightarrow[n\to\infty]{(d)} (\mathbb{e}_t; t \in [0,1]),$$

holds for the uniform topology, where $\mathbb{e}$ is the standard Brownian excursion with duration 1. Then this implies the convergence in distribution for the GH(P) topology of $\frac{1}{\sqrt{2n}} T^n$ to $\mathcal{T}_\mathbb{e}$ defined as in (1.2). The latter is called the standard Brownian Continuum Random Tree introduced by Aldous [Ald91a, Ald93].

The continuum analogue of the branching random walk on T^n on the Brownian CRT $\mathcal{T}_\mathbb{e}$ is as follows: conditionally on $\mathbb{e}$, let $Z^\mathbb{e}$ be a centred Gaussian random process with covariance function given for every $0 \le s \le t \le 1$ by

$$\mathbf{E}\left[Z^\mathbb{e}_s Z^\mathbb{e}_t \;\middle|\; \mathbb{e} \right] = \min_{[s,t]} \mathbb{e}.$$

One can check that indeed the function on the right is the covariance function of a Gaussian process and moreover that $Z^\mathbb{e}$ thus defined admits a modification that is (Hölder-)continuous, which is the one we shall implicitly use below. Moreover, if $d_\mathbb{e}(s,t) = 0$ (recall the definition from (1.1)), so s and t are identified in $\mathcal{T}_\mathbb{e}$, then $Z^\mathbb{e}_s = Z^\mathbb{e}_t$ a.s. so $Z^\mathbb{e}$ can be seen as being indexed by the tree $\mathcal{T}_\mathbb{e}$. Moreover $Z^\mathbb{e}_t$ has the conditional law of a centred Gaussian with variance $\mathbb{e}_t = d_\mathbb{e}(0,t)$ which is the length of the path to the root. Hence $Z^\mathbb{e}$ behaves on $\mathcal{T}_\mathbb{e}$ as a Brownian motion along each path, started at the root, which splits and continues independently at the branchpoints. The pair $(\mathbb{e}, Z^\mathbb{e})$ is called in the literature the head of the *Brownian snake*. The entire Brownian snake being the path-valued process that records for every time t the entire Brownian trajectory on the path from the root. See e.g. [DLG02, Chapter 4] for details on such processes in a broader setting.

The key starting point of the study of random quadrangulations is the convergence of the associated labelled trees proved by Chassaing & Schaeffer [CS04]. Let us also mention that Marckert & Mokkadem [MM03] proved that convergence of the head of the snakes is equivalent to convergence of the entire snakes.

Theorem 1.1 ([CS04]). *For the uniform distribution on well labelled trees, the convergence*

$$\left(\frac{1}{\sqrt{2n}}C^n_{2nt}, \left(\frac{9}{8n}\right)^{1/4} L^n_{2nt}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (\mathbb{e}_t, Z^{\mathbb{e}}_t)_{t\in[0,1]}$$

holds in distribution for the topology of uniform convergence.

1.1.4 The convergence of quadrangulations

Let us derive some consequences of Theorem 1.1. Let Q^n denote a uniformly random quadrangulation with n faces and $v^n_\star$ a distinguished vertex sampled independently uniformly at random among the $n+2$ possibilities. We shall denote the graph distance in Q^n by d_{Q^n}. Let (T^n, ℓ^n) be the associated uniformly random well labelled tree, coded by the pair of paths (C^n, L^n). See Figure 1.5 for simulations of a large random labelled tree and the corresponding quadrangulation.

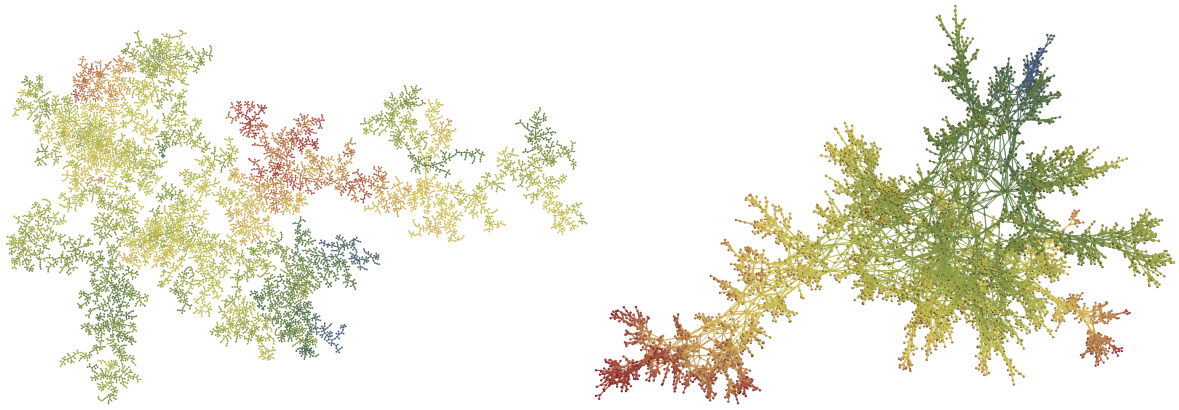


Figure 1.5 – Left: a random well labelled tree with $n = 10\,000$ edges with labels indicated by colours. Right: the associated pointed quadrangulation, non isometrically embedded in the space, where colours indicate the graph distance to the distinguished vertex (blue = closer).

Profile of distances

Recall that the labels code the distances to $v^n_\star$ in Q^n, namely, if $c^n_0, \dots, c^n_{2n}$ is the contour sequence of T^n as previously, and if we abuse notation and write c^n_i for the vertex in Q^n which corresponds to the vertex of T^n incident to the corner c^n_i, then for every $0 \le i \le 2n$,

$$d_{Q^n}(c^n_i, v^n_\star) = L^n_i - \min L^n + 1. \tag{1.3}$$

Theorem 1.1 then immediately implies the following convergences in distribution:

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q^n}(c^n_0, v^n_\star) \xrightarrow[n\to\infty]{(d)} -\min Z^{\mathbb{e}} \qquad \text{and} \qquad \left(\frac{9}{8n}\right)^{1/4} \max_{v\in V(Q^n)} d_{Q^n}(v, v^n_\star) \xrightarrow[n\to\infty]{(d)} \max Z^{\mathbb{e}} - \min Z^{\mathbb{e}}.$$

By construction, c^n_0 is one of the extremities of the root edge of Q^n.

We can also deduce the convergence of the so-called *two-point function* of Q^n, namely that if u^n and v^n are two independent uniform random vertices sampled independently of Q^n, then

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q^n}(u^n, v^n) \xrightarrow[n\to\infty]{(d)} Z^{\mathbb{e}}_U - \min Z^{\mathbb{e}}, \tag{1.4}$$

where U has the uniform distribution on $[0,1]$ and is independent of $Z^{\mathbf{e}}$. Indeed, one can take $v^n = v^n_\star$, but beware that $c^n_{\lfloor 2Un \rfloor}$ does not have the uniform distribution on the vertices since the sequence of corners contains redundancies. We solve this issue with the same trick as at the very end of Section 1.1.3.

This convergence of the two-point function can alternatively be rewritten as that of the so-called *profile of distances*: for every $k \geq 0$, let

$$I_{Q^n, v^n_\star}(k) = \frac{1}{n+2} \cdot \#\{v \in V(Q^n) : \ d_{Q^n}(v, v^n_\star) = k\}$$

denote the proportion of vertices at distance k from the distinguished vertex. Let us view $I_{Q^n, v^n_\star}(\cdot)$ as a probability measure, then we deduce from (1.4) the convergence in distribution for the topology of weak convergence of measures

$$I_{Q^n, v^n_\star}\left(\left(\frac{9}{8n}\right)^{1/4} \cdot\right) \ \xrightarrow[n\to\infty]{(d)} \ I$$

where I is defined for every continuous and bounded function g by

$$\int g \, \mathrm{d}I = \int_0^1 g(Z^{\mathbf{e}}_t - \min Z^{\mathbf{e}}) \, \mathrm{d}t,$$

and is the called the *occupation measure* of $Z^{\mathbf{e}}$ above its infimum.

Metric space convergence

Using (1.3) and the triangle inequality at $v^n_\star$, one can upper bound the distance between any two vertices of Q^n using the label process. By being more clever and considering the first point at which two geodesics to $v^n_\star$ touch each other, Le Gall [LG07, Lemma 3.1] obtained the following upper bound: for every $0 \leq i \leq j \leq 2n$,

$$d_{Q^n}(c^n_i, c^n_j) \leq D^n(i,j) = L^n_s + L^n_j - 2\max\left\{\min_{k\in[i,j]} L^n_k; \min_{k\in[0,i]\cup[j,2n]} L^n_k\right\} + 2. \tag{1.5}$$

For any continuous function $g : \ [0,1] \longrightarrow \mathbf{R}$, let us set for every $0 \leq s \leq t \leq 1$,

$$D_g(s,t) = D_g(t,s) = g(s) + g(t) - 2\max\left\{\min_{r\in[s,t]} g(r); \min_{r\in[0,s]\cup[t,1]} g(r)\right\}.$$

By continuity, Theorem 1.1 implies that, for the topology of uniform convergence,

$$\left(\frac{9}{8n}\right)^{1/4} D^n(\lfloor 2n\cdot \rfloor, \lfloor 2n\cdot \rfloor) \ \xrightarrow[n\to\infty]{(d)} \ D_{Z^{\mathbf{e}}}(\cdot,\cdot).$$

From here one can deduce tightness of the rescaled distance function and thus the convergence for the topology of uniform convergence *along subsequences*:

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q^n}\left(c^n_{\lfloor 2n\cdot \rfloor}, c^n_{\lfloor 2n\cdot \rfloor}\right) \ \xrightarrow[n\to\infty]{(d)} \ D_\infty(\cdot,\cdot), \tag{1.6}$$

where D_∞ is a continuous pseudo-distance which depends a priori on the subsequence and, by (1.5), satisfies $D_\infty \leq D_{Z^{\mathbf{e}}}$, see [LG07, Proposition 3.2].

Viewing Q^n as a metric space, this implies the convergence along the same subsequence

$$\left(\frac{9}{8n}\right)^{1/4} Q^n \ \xrightarrow[n\to\infty]{(d)} \ M_\infty = [0,1] \big/ \{D_\infty = 0\}$$

in the GH topology. In order to extend it to a convergence in the GHP topology, as mentioned earlier for trees, one needs to remove redundancies in the list of vertices $c^n_0, \ldots, c^n_{2n}$ and obtain the analogue of (1.6) for such a sequence, which can be done as previously.

In order to be able to conclude to the convergence of the maps without extraction, the major issue is then to prove that all the subsequential limits D_∞ have the same law, say as some D^*, which can be defined directly from the pair $(\mathfrak{e}, Z^{\mathfrak{e}})$. In the case of random quadrangulations, this was solved simultaneously in [LG13, Mie13] with D^* the largest pseudo-distance D such that for every $s, t \in [0,1]$, it holds

$$D(s,t) \le D_{Z^{\mathfrak{e}}}(s,t) \qquad \text{and} \qquad D(s,t) = 0 \ \text{ as soon as } \ d_{\mathfrak{e}}(s,t) = 0. \tag{1.7}$$

The space

$$\mathcal{S} = [0,1] \big/ \{D^* = 0\}$$

is called the *Brownian sphere*.

Theorem 1.2 ([LG13, Mie13])**.** *Let Q^n denote a quadrangulation of the sphere with n faces sampled uniformly at random. Then the convergence in distribution*

$$\left(\frac{9}{8n}\right)^{1/4} Q^n \ \xrightarrow[n\to\infty]{(d)} \ \mathcal{S}$$

holds in the Gromov–Hausdorff(–Prokhorov) topology.

The rerooting trick

Let us briefly explain a clever trick due to Le Gall [LG13] that will allow us to *deduce* from Theorem 1.2 more general invariance principles in the next sections.

It should be clear from the previous discussion that deducing the convergence in distribution of quadrangulations along subsequences from the convergence in distribution of the associated random paths to $(\mathfrak{e}, Z^{\mathfrak{e}})$ is rather straightforward, and moreover the subsequential limits D_∞ are pseudo-distances satisfying (1.7). Let U and V be independent uniform random variables in $[0,1]$ independent of the rest. According to (1.4), or rather the three lines below, we have

$$D_\infty(U,V) \ \overset{(d)}{=} \ Z^{\mathfrak{e}}_U - \min Z^{\mathfrak{e}}. \tag{1.8}$$

Now let us suppose that, for another model of random maps, we arrive at the same conclusion, with subsequential limits D'_∞ satisfying (1.7) and (1.8). Then we can deduce immediately that $D'_\infty = D^*$ effortless! Indeed, thanks to Theorem 1.2, we have $D_\infty = D^*$ and so

$$D^*(U,V) \ \overset{(d)}{=} \ Z^{\mathfrak{e}}_U - \min Z^{\mathfrak{e}}. \tag{1.9}$$

Moreover, by (1.7) and the maximality property of D^*, we have $D'_\infty \le D^*$ almost surely and thus $D'_\infty(U,V)$ and $D^*(U,V)$ not only agree in law but almost surely. By sampling a sequence of i.i.d. uniform random times, the almost sure equality $D'_\infty = D^*$ then follows by a density argument.

Remark 1.2. All this trick relies on the identification of the law in (1.9) that one would believe is provable directly in the continuum setting but the only known proofs [LG13, Mie13] use the discrete approximation by quadrangulations, namely they work hard to prove that $D_\infty = D^*$ and then deduce (1.9) from (1.8). An independent proof of (1.9) would certainly be very much appreciated!

This rerooting trick introduced in [LG13] let Le Gall proved there the convergence to the Brownian sphere of random triangulations. Since then it has been used to prove the convergence of more models of maps to the Brownian sphere [BLG13, BJM14, Abr16, ABA17, ABA21], and our work is not an exception [Mar18b, Mar18a, Mar22a, Mar22b].

Quadrangulations with other topologies

Let us end this section with a few words on other model of quadrangulations, on more general surfaces. A first extension is to consider quadrangulations *with a boundary*, i.e. with only faces of degree 4, except the root face which has an arbitrary even degree. Let Q^{n,ϱ_n} denote such a uniformly chosen random quadrangulation with boundary length $2\varrho_n$ and n inner quadrangular faces. Bettinelli [Bet15] studied this model, proved the convergence of the associated random labelled trees in an extension of the CVS correspondence, and then the convergence of these maps along subsequences. The uniqueness of the limit was later proved by Bettinelli & Miermont [BM17], which allows to use the rerooting trick exactly as above in the next section, based on the following theorem.

Theorem 1.3 ([Bet15, BM17]). *Let $\varrho \in [0,\infty)$ and $(\varrho_n)_{n\geq 1}$, $(\varrho'_n)_{n\geq 1}$ be sequences such that $(2n)^{-1/2}\varrho_n \longrightarrow \varrho$ and $(2n)^{-1/2}\varrho'_n \longrightarrow \infty$. Then the convergences in distribution*

$$\left(\frac{9}{8n}\right)^{1/4} Q^{n,\varrho_n} \xrightarrow[n\to\infty]{(d)} \mathcal{S}^{\varrho} \qquad \text{and} \qquad \frac{1}{\sqrt{2\varrho'_n}} Q^{n,\varrho'_n} \xrightarrow[n\to\infty]{(d)} \mathcal{T}_{\mathbb{e}}$$

hold in the Gromov–Hausdorff topology, where $\mathcal{S}^0$ is the Brownian sphere and $\mathcal{S}^{\varrho}$ is called the Brownian disk *with perimeter $\varrho > 0$ otherwise, and $\mathcal{T}_{\mathbb{e}}$ is the standard Brownian CRT coded by $\mathbb{e}$.*

As for more general surfaces, the CVS bijection has been extended by Chapuy, Marcus, & Schaeffer [CMS09] to apply to bipartite quadrangulations on a surface with any fixed genus $g \geq 1$. This allowed exactly as above Chapuy [Cha10] to control the radius and profile of the distances and Bettinelli [Bet10, Bet12, Bet16] to prove the GH convergence along subsequences, compute the Hausdorff dimension of these limits, and study their topology and geometry. To this day, showing uniqueness of the subsequential limits is not yet complete, but investigated by Bettinelli & Miermont. We shall not at all try to generalise these results here, but this would be a natural direction of future research.

1.2 Random maps with prescribed face degrees

We describe in this section the papers [Mar18b, Mar22a, Mar22b] on a model of uniform random (bipartite, plane) maps with prescribed face degrees and the convergence of such maps following the recipe described in the previous section. Let us recall that the results from [Mar18b] are entirely generalised in [Mar22a].

1.2.1 Model and main results

Let us consider for every $n \geq 1$ an integer $\varrho_n \geq 1$ and a finite sequence of integers $\mathbf{f}_n = (\mathrm{f}_{n,i})_{1\leq i\leq n}$ such that $\mathrm{f}_{n,1} \geq \cdots \geq \mathrm{f}_{n,n} \geq 1$, and let $M^{\varrho_n,\mathbf{f}_n}$ denote a uniformly chosen (rooted plane) map with perimeter $2\varrho_n$ and n inner faces with degrees given by the $2\mathrm{f}_{n,i}$'s. Note that $M^{\varrho_n,\mathbf{f}_n}$ always has $n+1$ faces in total and

$$E_n := \varrho_n + \sum_{i=1}^{n} \mathrm{f}_{n,i} \quad \text{edges, and so} \quad V_n := E_n + 2 - (n+1) = 1 + \varrho_n + \sum_{i=1}^{n} (\mathrm{f}_{n,i} - 1) \quad \text{vertices}$$

by Euler's formula. In order to avoid trivialities, we shall assume that $V_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Next, as in the case of quadrangulations, it will be simpler to work with pointed maps, having a distinguished vertex $v^n_\star$. Since the number of vertices is fixed, this is equivalent to first sampling the non-pointed map $M^{\varrho_n,\mathbf{f}_n}$ uniformly at random and then distinguishing one of its V_n vertices independently and uniformly at random. The fundamental quantity which appears in our statements is a kind of second moment factorial:

$$\sigma_n^2 = \sum_{i=1}^{n} \mathrm{f}_{n,i}(\mathrm{f}_{n,i} - 1). \tag{1.10}$$

The reader may question this definition of σ_n^2 and e.g. replace it by a variance. In Section 1.2.4 it will appear naturally in relation with the mean degree minus one under the size-biased distribution.

The first general result, excerpt from [Mar22a], is the identification of the growth of these maps, which is $\sigma_n^{1/2} = (\sigma_n^2)^{1/4}$, unless the boundary is very large, in which case it takes over the rest.

Theorem 1.4 ([Mar22a]). *From every increasing sequence of integers, one can extract a subsequence along which $(\varrho_n + \sigma_n)^{-1/2} M^{\varrho_n, \mathbf{f}_n}$ converges in distribution in the GHP topology to a limit with a nonzero diameter.*

The strength of this result is that it makes no assumption at all on the degrees. Of course, nothing can be said about the subsequential limits at this level of generality. This is done in [Mar22a] in two extreme regimes: in the case of no macroscopic inner face and in the case of a very large boundary.

Theorem 1.5 ([Mar22a]). *Theorem 1.4 can be completed in two extreme regimes:*

(i) Suppose that $\sigma_n^{-1}\varrho_n \longrightarrow \varrho \in [0, \infty)$, and that $\sigma_n^{-1}\mathrm{f}_{n,1} \longrightarrow 0$. Then the convergence in distribution

$$\sqrt{\frac{3}{2\sigma_n}} \cdot M^{\varrho_n, \mathbf{f}_n} \xrightarrow[n\to\infty]{(d)} \mathcal{S}^{\varrho}$$

holds in the GHP topology, where $\mathcal{S}^{\varrho}$ is the Brownian sphere when $\varrho = 0$ and the Brownian disk with perimeter $\varrho > 0$ otherwise.

(ii) Suppose that $\sigma_n^{-1}\varrho_n \longrightarrow \infty$, then the convergence in distribution

$$\frac{1}{\sqrt{2\varrho_n}} \cdot M^{\varrho_n, \mathbf{f}_n} \xrightarrow[n\to\infty]{(d)} \mathcal{T}_{\mathbb{e}}$$

holds in the GHP topology, where $\mathcal{T}_{\mathbb{e}}$ is the Brownian CRT coded by the Brownian excursion $\mathbb{e}$.

In the case $\varrho_n = 1$, or equivalently for maps without boundary, the convergence to the Brownian sphere was proved previously in [Mar18b] in a restricted regime, in which σ_n^2 is of order n, so the scaling is of the usual order $n^{1/4}$. Although the statements extend the case of quadrangulations [LG13, Mie13, BM17], the convergence to the Brownian sphere or disk strongly relies on these works via the rerooting trick discussed in Section 1.1.4.

Recall that for an integer $p \geq 2$, a *$2p$-angulation* denotes a map in which all faces have degree $2p$; in this case $\sigma_n^2 = p(p-1)n$. Le Gall [LG13, Theorem 1] proved that for any p fixed, such a $2p$-angulation with n faces sampled uniformly at random converges in distribution towards the Brownian sphere; this was extended to $2p$-angulations with a boundary by Bettinelli & Miermont [BM17, Corollary 6]. Theorem 1.5 allows to let p vary with n.

Corollary 1.1 ([Mar22a]). *Let $\varrho \in [0, \infty]$ and let $(p_n)_{n\geq 1} \in \{2, 3, \ldots\}^{\mathbb{N}}$ and $(\varrho_n)_{n\geq 1} \in \mathbb{N}^{\mathbb{N}}$ be any sequences such that $\lim_{n\to\infty}(p_n(p_n-1)n)^{-1/2}\varrho_n = \varrho$. For every $n \geq 1$, let $M^n_{\varrho_n, p_n}$ be a uniformly chosen random $2p_n$-angulation with n inner faces and with perimeter $2\varrho_n$.*

(i) If $\varrho < \infty$, then $(\frac{4}{9}p_n(p_n-1)n)^{-1/4} M^n_{\varrho_n, p_n} \longrightarrow \mathcal{S}^{\varrho}$ in distribution for the GHP topology.

(ii) If $\varrho = \infty$, then $(2\varrho_n)^{-1/2} M^n_{\varrho_n, p_n} \longrightarrow \mathcal{T}_{\mathbb{e}}$ in distribution for the GHP topology.

Theorem 1.5 leaves open the question of the behaviour of the maps in presence of macroscopic faces, with degree of order σ_n. This is considered in [Mar22b] in which we assume that there exists a sequence $\boldsymbol{\theta}$ of real numbers $\theta_1 \geq \theta_2 \geq \cdots \geq 0$ such that $\sigma_n^{-1}\mathrm{f}_{n,i} \longrightarrow \theta_i$ for every $i \geq 1$. By Fatou's lemma, we have $\sum_{i\geq 1} \theta_i^2 \leq 1$; let us define $\theta_0 \in [0, 1]$ by $\theta_0^2 = 1 - \sum_{i\geq 1} \theta_i^2$. Let us refer to [AHUB20b, Section 6.2] for explicit examples of such triangular arrays in the case $\theta_0 = 0$ and $\sum_i \theta_i = \infty$ or in the case $\theta_0, \theta_1, \theta_2, \ldots > 0$.

Recall that $v_\star^n$ denotes a vertex of $M^{\varrho_n, \mathbf{f}_n}$ sampled independently and uniformly at random. We let $d_{M^{\varrho_n, \mathbf{f}_n}}$ denote the graph distance in $M^{\varrho_n, \mathbf{f}_n}$ and $d_{M^{\varrho_n, \mathbf{f}_n}}(v_\star^n, \vec{e}_n)$ denote the smallest distance between the vertex $v_\star^n$ and the endpoints of the root edge.

Theorem 1.6 ([Mar22b]). *Suppose that $\sigma_n^2 \longrightarrow \infty$, and that there exist $\varrho \in [0,\infty)$ and a sequence $\boldsymbol{\theta}$ such that*

$$\sigma_n^{-1}\varrho_n \longrightarrow \varrho, \qquad \text{and} \qquad \sigma_n^{-1}\mathrm{f}_{n,i} \longrightarrow \theta_i \quad \text{for every } i \geq 1.$$

Then there exists a continuous process $Z^{\varrho,\boldsymbol{\theta}} = (Z_t^{\varrho,\boldsymbol{\theta}}; t \in [0,1])$, whose law only depends on ϱ and $\boldsymbol{\theta}$ such that the following holds:

(i) We have

$$\frac{d_{M^{\varrho_n,\mathrm{f}_n}}(v_\star^n, \vec{e}_n)}{\sqrt{2\sigma_n}} \xrightarrow[n\to\infty]{(d)} -\min Z^{\varrho,\boldsymbol{\theta}} \qquad \text{and} \qquad \max_{v\in V(M^{\varrho_n,\mathrm{f}_n})} \frac{d_{M^{\varrho_n,\mathrm{f}_n}}(v_\star^n, v)}{\sqrt{2\sigma_n}} \xrightarrow[n\to\infty]{(d)} \max Z^{\varrho,\boldsymbol{\theta}} - \min Z^{\varrho,\boldsymbol{\theta}}.$$

(ii) The random probability measure

$$I^n(\mathrm{d}x) = \frac{1}{V_n} \cdot \#\left\{ v \in V(M^{\varrho_n,\mathrm{f}_n}) : \frac{d_{M^{\varrho_n,\mathrm{f}_n}}(v_\star^n, v)}{\sqrt{2\sigma_n}} \in \mathrm{d}x \right\}$$

converges in distribution for the topology of weak convergence of measures to the occupation measure $I^{\varrho,\boldsymbol{\theta}}$ of $Z^{\varrho,\boldsymbol{\theta}}$ above its infimum, which is the random measure defined for every continuous and bounded function g by

$$\int g \,\mathrm{d}I^{\varrho,\boldsymbol{\theta}} = \int_0^1 g(Z_t^{\varrho,\boldsymbol{\theta}} - \min Z^{\varrho,\boldsymbol{\theta}})\,\mathrm{d}t.$$

(iii) The subsequential limits of $(2\sigma_n)^{-1/2} M^{\varrho_n,\mathrm{f}_n}$ provided by Theorem 1.4 take the form of a quotient space $[0,1]/\{D_\infty = 0\}$ where D_∞ is a random continuous pseudo-distance on $[0,1]$ such that if U is sampled uniformly at random on $[0,1]$ and independently of the rest, then

$$D_\infty(U,\cdot) \overset{(d)}{=} Z^{\varrho,\boldsymbol{\theta}} - \min Z^{\varrho,\boldsymbol{\theta}}.$$

The proof of these results follows the general guideline of Le Gall [LG13] described for quadrangulations in the previous section, namely:

Step i: Code the map by a labelled tree such that the labels describe the distances to $v_\star^n$ in the map.

Step ii: Code this labelled tree by a pair of paths.

Step iii: Prove that the paths are always tight when suitably normalised, this leads to Theorem 1.4.

Step iv: Under appropriate assumptions on the face degrees, prove that the paths converge in distribution, this leads to Theorem 1.6,

Step v: Use the rerooting trick to deduce the convergence to a Brownian disk in the case $\mathrm{f}_{n,1} \ll \sigma_n$; the tree regime $\sigma_n \ll \varrho_n$ is simpler, this leads to Theorem 1.5.

In the rest of this section we describe the first four steps, up to the convergence of the label process, the rest follows as in the previous section. We shall restrict to maps without boundary to simplify the exposition: the case of a boundary does not change much but adds more notation.

1.2.2 Bipartite maps as labelled looptrees

The proofs of scaling limits of random maps following the previous method often use in Step i the extension of the CVS bijection constructed by Bouttier, Di Francesco, & Guitter [BDFG04]; this is the case in e.g. [LG13, Abr16, BM17, ABA21] where notably the last paper is not restricted to bipartite maps. This bijection relates any plane map with more involved labelled trees called *mobiles*. They take a simpler form in the case of bipartite maps, but still more complicated than in the CVS bijection. Later,

Janson & Stefánsson [JS15] constructed a bijection between these mobiles associated with bipartite maps and simpler labelled trees. These are the labelled trees used in [Mar18b, Mar18a, Mar22a]. The last work [Mar22a] has something special in the sense that it is the first one where a strong convergence of maps is obtained without a strong control of the geometry of the trees associated with them: indeed the diameter (or maximal height) of these trees is not known in general! This suggests that the trees are not the correct object to consider. Recently, in [Mar22b], we thus proposed a new formulation which in some sense unifies the mobiles of [BDFG04] whose construction is natural, and the trees of [JS15] which are simpler to study. We did so by considering (a variant of) the so-called *looptrees* introduced by Curien & Kortchemski [CK14] which we now present.

Labelled looptrees and bipartite maps

The concept of looptrees was introduced in details in [CK14] and successfully used in relation with maps [DMS21, MS21a, CK15, KR19, KR20, LG18, Ric18, SS19, BHS18], but also for their own interest or in relation with other models [Arc20a, Arc21, AS21, BS15, CDKM15, CHK15]. Our version slightly differs from that of [CK14] and already appeared (without labels) in [CK15, Ric18] and can be defined as follows.

Definition 1.1. A *looptree* is a plane map which satisfies the property that there is a distinguished outer face, to the left of the root edge such that each edge has exactly one side incident to this face.

Note that the definition implies that all the inner faces are simple cycles and are edge-disjoint; also no edge is pending inside the outer face. The looptrees from [CK14] ironically forbid loops as well as vertices with degree more than 4. The term looptree is better explained in a picture and looptrees are in one-to-one correspondence with plane trees, see Figure 1.6.

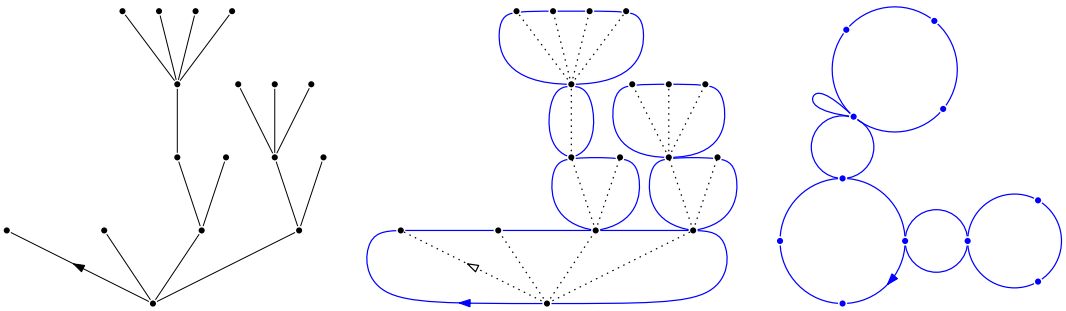


Figure 1.6 – From left to right: A plane tree, its looptree version as defined in [CK14], and the version we consider here obtained by further merging each internal vertex of the tree with its right-most offspring.

We shall equip looptrees with labels that assign to each vertex an integer, not necessarily distinct, see the right of Figure 1.9 for an example. We say that a labelling of a looptree is a *good labelling* (in which case the looptree is *well labelled*) if the label increment along each edge oriented by keeping the outer face to its left lies in $\mathbf{Z}_{\geq -1} = \{-1, 0, 1, 2, \dots\}$. Note that the sum of these increments over all the edges on the same cycle must vanish, so for each cycle with length say $k \geq 1$, the vector of label increments read in clockwise order belongs to the set

$$\mathbf{B}_k^{\geq -1} := \left\{ (x_1, \dots, x_k) \in \mathbf{Z}_{\geq -1}^k : \; x_1 + x_2 + \cdots + x_k = 0 \right\}. \tag{1.11}$$

The labelling of the vertices is determined by the increments along the edges up to a global shift, which we fix by setting the root label to 0.

The next result is really just a reformulation of the bijection from [BDFG04].

Lemma 1.1 ([BDFG04, Mar22b]). *There is a two-to-one correspondence between pointed bipartite plane maps and well labelled looptrees, which enjoys the following properties:*

(i) The cycles of the looptree correspond to the faces of the map, and the length of a cycle is half the degree of the associated face;

(ii) The vertices of the looptree correspond to the non-distinguished vertices of the map, and their label, minus the smallest label plus one, equals the graph distance in the map of the associated vertex to the distinguished one.

As in the CVS correspondence, the only information on maps which is lost in the looptrees is the orientation of the root edge. Note that by Euler's formula, the looptree and the map have the same amount of edges.

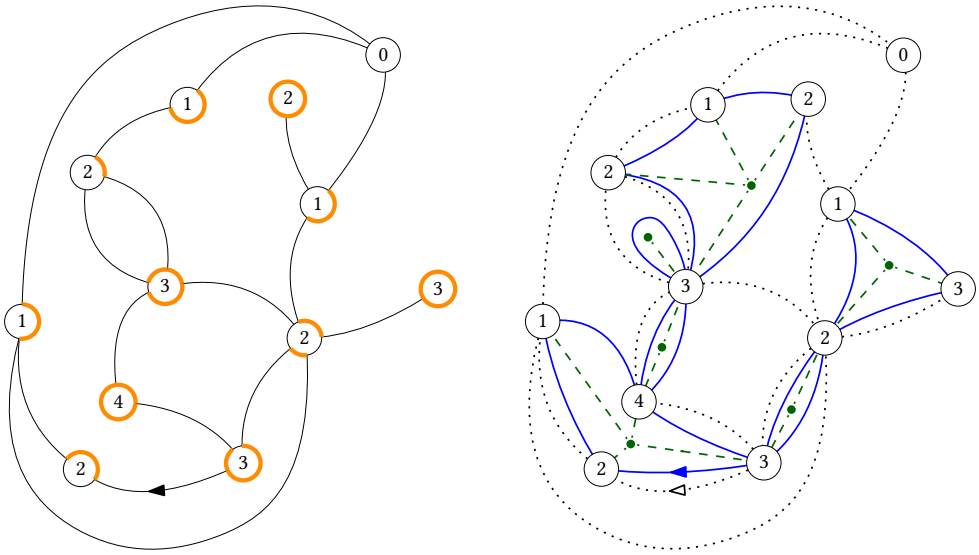


Figure 1.7 – *Left:* a map with its vertices labelled by their distance to a distinguished vertex and its marked corners in orange. *Right:* the corresponding mobile from [BDFG04] in dashed green, obtained by linking each marked corner in a face to an extra vertex inside, and in plain blue the corresponding looptree, obtained instead by joining the marked corners in cycles.

The construction of the labelled looptree from the pointed map goes as follows. First label all the vertices by their distance to the distinguished vertex. Then in each face, mark each corner if the next one in clockwise order inside the face has a smaller label. Note that in a bipartite map, the labels along each edge must vary by ±1 so half of the corners in each face are marked. Also each vertex of the map except the distinguished one has at least one marked corner. In [BDFG04] the mobile is constructed by adding a new vertex inside each face and linking it to each marked corner of that face; here instead we join the marked corners together in a cycle.

The construction of the map from the labelled looptree is the exact analogue of Schaeffer's construction of quadrangulations. Indeed, let us shift all labels so the minimum is 1. Then follow the outer face in clockwise order and link each external corner of the looptree to the next one (in the infinite periodic sequence) with a smaller label, which in fact can only be smaller by exactly 1. This construction fails for the corners labelled 1, so instead we join them all to an extra vertex labelled 0 in the outer face. The edge emanating from the root corner of the looptree is chosen as the root edge of the map.

Remark 1.3. When all the faces of the map are quadrangles, then all the cycles of the looptrees have length 2, so a good labelling can actually only vary by either -1, 0, or $+1$ along each oriented edge.

This construction in fact reduces exactly to Schaeffer's bijection in which each edge of the tree has been doubled, see Figure 1.8 for an example. In this case the relation between the looptree and its corresponding tree as in Figure 1.6 is a classical bijection between plane trees and binary plane trees.

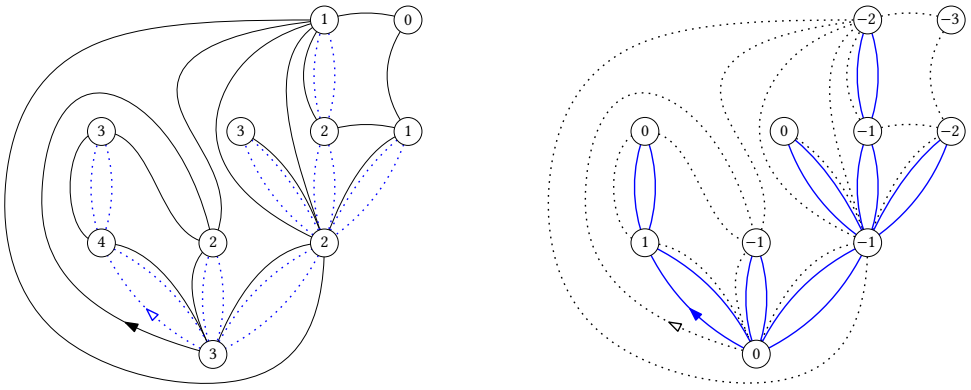


Figure 1.8 – In the particular case of quadrangulations, the looptree reduces to Schaeffer's tree in which each edge has been doubled (compare with Figure 1.3).

Lemma 1.1 unifies in a sense the constructions of [BDFG04] and [JS15]. Indeed, the former draws a correspondence between maps and labelled *mobiles*, which are trees in which two types of vertices alternate: unlabelled black ones and labelled white ones, corresponding respectively to the faces and non-distinguished vertices of the map. This mobile is simply a 'vertex-dual' graph of the looptree, obtained by linking each (white) vertex in each cycle to an extra black vertex inside. Then Janson & Stefánsson [JS15] built a correspondence between these mobiles and trees, which sends the black and white vertices to the internal vertices and leaves respectively. One can check from their construction, also discussed in [Mar18b, Section 2.4], that the looptree associated with the JS-tree in the sense of Figure 1.6 exactly corresponds to the looptree associated with the map in Lemma 1.1, see Figure 1.9.

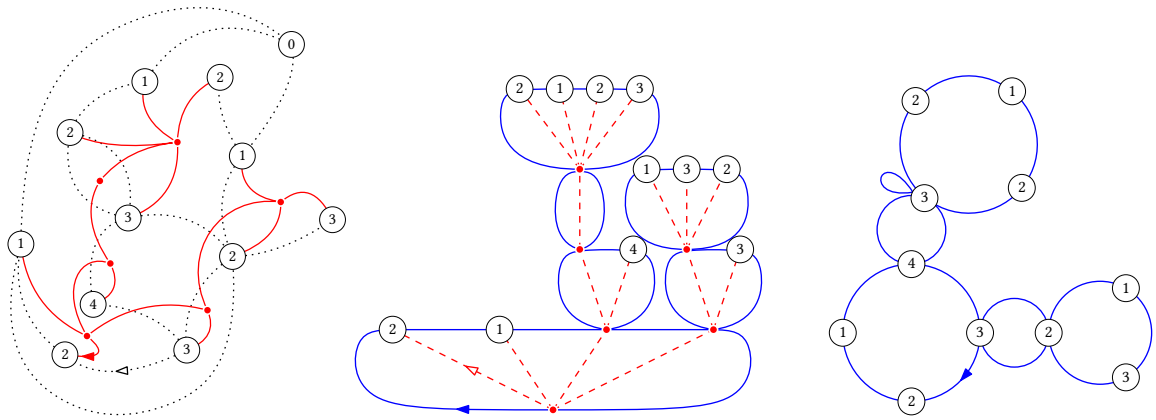


Figure 1.9 – *Left:* The map in dotted lines and the labelled tree obtained by combining [BDFG04] and [JS15] in plain red. *Middle:* The same tree in dashed red and in plain blue its looptree version as in [CK14]. *Right:* The looptree version we consider here.

Coding paths

Let us move on to Step II in the proof scheme of the convergence of maps, which consists in encoding the labelled looptree by a pair of paths, one describing its geometry and the other one the labels, as illustrated in Figure 1.10. The main reason why we rewrote the bijection from [BDFG04] in terms of

looptrees is that their geometry is explicitly encoded by the so-called *Łukasiewicz path* of the associated tree, whereas the geometry of the tree involves the much more complicated *height* or *contour* process. We chose to be very precise here in defining the paths in the discrete setting, let us already warn the reader that this will not be the case in the continuum setting. More details can also be found e.g. in [LG05].

The Łukasiewicz path Let T denote a (rooted plane) tree with $N+1$ vertices, which we see as encoding the genealogy of a family, seeing vertices as individuals. From this point of view, the root vertex is called the ancestor of the family, for every other vertex, its neighbour closest to the ancestor is its parent, while the other neighbours are its children; finally a vertex with no child is called a leaf and the other vertices are said to be internal. The fact that the tree is embedded on the plane is equivalent to an ordering from left to right of the children of every internal vertex; the root edge points canonically to the leftmost child of the ancestor.

The local ordering of the vertices, together with the starting point given by the root edge, allows to order in several canonical ways the vertices of the tree. The Ulam–Harris–Neveu formalism consists in coding a tree by a *word* $T \in \mathbf{U} = \bigcup_{n\geq 0} \mathbf{N}^n$, where $\mathbf{N} = \{1, 2, \dots\}$ and $N^0 = \{\emptyset\}$ as follows:

(i) the root of the tree is $\emptyset$;

(ii) if $u = u_1 \cdots u_g \in \mathbf{U}$ belongs to the tree and has $k(u) \in \mathbf{N}$ children, then these are given by $uj = u_1 \dots u_g\, j$ for $1 \leq j \leq k(u)$ from left to right.

For any vertex $u = u_1 \cdots u_g$ in a tree, we shall write $|u| = g$ for its generation. From this formalism, one can order the vertices of a tree T with $N+1$ vertices in a canonical way, say $v_0, \dots, v_N$, by using the lexicographical order on $\mathbf{U}$. This ordering corresponds to the depth-first search order, which can be defined via the following algorithm:

(i) let c_0 denote the root vertex;

(ii) if the vertices $c_0, \dots, c_i$ have been constructed for some $i \in \{0, \dots, 2N-1\}$, and $c_i = u_1 \cdots u_g$, then let c_{i+1} be the leftmost child of c_i which does not belong to $(c_0, \dots, c_i)$ if any, and let it be $u_1 \cdots u_{g-1}$ the parent of c_i otherwise.

The sequence $c_0, \dots, c_{2N}$ can actually be seen as the ordering of the *corners* of the tree (note that the first and last corner coincide), following its contour. Each vertex appears in this list as many times as its degree and retaining only the first appearance of each gives back the sequence $v_0, \dots, v_N$ in the lexicographical order.

The Łukasiewicz path $W = (W_j; 0 \leq j \leq N+1)$ is then defined by $W_0 = 0$ and

$$W_j = \sum_{i=0}^{j-1} (k(v_i) - 1)$$

for $1 \leq j \leq N+1$. One can note that $W_j \geq 0$ for every $j \leq N$ but $W_{N+1} = -1$, and $\Delta W_j := W_{j+1} - W_j \geq -1$ for all $j \leq N$. We shall always implicitly extend W by adding flat steps between integer times. One can recover the tree from its Łukasiewicz path; indeed consider a nonnegative increment, say $\Delta W_i = k \geq 0$. Then the vertex v_i of T has $k+1$ children, which are given by $v_{j_1}, \dots, v_{j_{k+1}}$ satisfying for each $\ell \in \{1, \dots, k+1\}$,

$$j_\ell = \inf\{j \geq i+1 : \ W_j = W_{i+1} - \ell + 1\}. \tag{1.12}$$

In particular $j_1 = i+1$ and $j_k = \inf\{j \geq i+1 : \ W_j = W_i\}$.

Let LT denote the looptree version of T in the sense of Figure 1.6. Note that:

- LT has N edges as well,

- its vertices are the leaves of T, which correspond to the negative increments of W,
- its cycles correspond to the internal vertices of T, matching the cycle lengths with the offspring numbers, which equal the size plus one of the nonnegative increments of W.

We can thus describe the Łukasiewicz path directly in terms of the looptree as follows. Following the outer face of LT by keeping it to the left defines a sequence $(e_0, \dots, e_N)$ of oriented edges, starting and ending at $e_0 = e_N$ the root edge of the looptree. For every $i \in \{0, \dots, N\}$, let $|e_i|$ denote the length of the cycle adjacent to e_i *if* e_i is the first edge in $(e_j)_{j \geq 0}$ adjacent to this cycle, and let $|e_i| = 0$ otherwise, then

$$|e_i| = k(v_i) \qquad \text{so} \qquad W_j = \sum_{i=0}^{j-1} (|e_i| - 1)$$

for $1 \leq j \leq N+1$. In the other direction, each nonnegative increment $\Delta W_i = k \geq 0$ of the Łukasiewicz path codes a cycle of looptree, given by $v_i, v_{j_1}, \dots, v_{j_{k+1}}$ in the notation from (1.12), where we recall that $v_{j_{k+1}}$ is identified with v_i in the looptree. Moreover these vertices are the origin of the edges $e_{j_1}, \dots, e_{j_{k+1}}$ respectively when following the contour of the looptree. Note that for every such vertex v_{j_ℓ}, there is a notion of left and right part of the loop, given by the vertices $v_i, v_{j_1}, \dots, v_{j_\ell}$ and $v_{j_\ell}, \dots, v_{j_{k+1}}$ respectively.

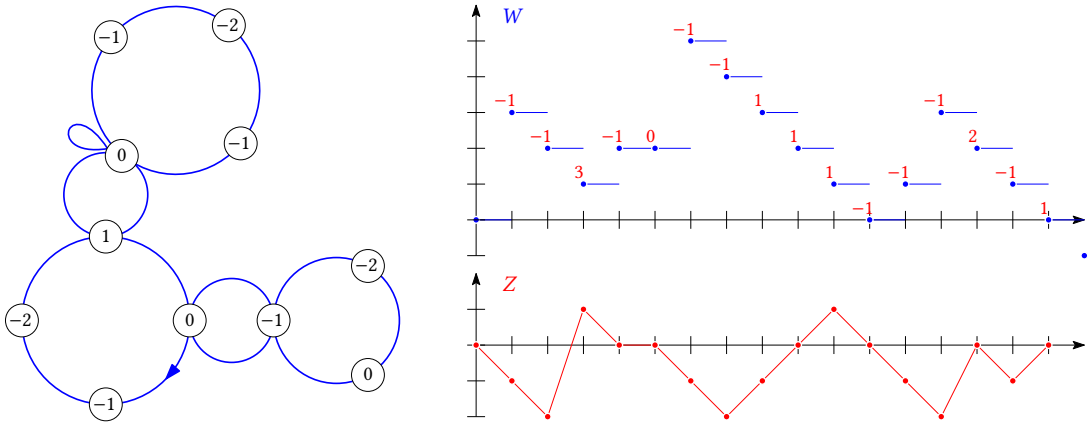


Figure 1.10 – *Left:* A well labelled looptree. *Top right:* its Łukasiewicz path W in blue decorated by the values of the label increments along each edge in red, viewed as being indexed by the jumps of W, from top to bottom. *Bottom right:* the label process Z, which equals the cumulative sum of the decorations on the Łukasiewicz path.

The label process Next suppose that the looptree is labelled, meaning that every vertex carries a number. Recall the sequence $(e_0, \dots, e_N)$ of oriented edges when following its contour, for each $0 \leq i \leq N$, let c_i denote the external corner at the origin of e_i and denote by c_i as well its incident vertex. Then we define the *label process* $Z = (Z_i)_{0 \leq i \leq N}$ such that Z_i is the label of c_i for every $0 \leq i \leq N$. Since $c_N = c_0$, then $Z_0 = Z_N$ which shall always be 0. We further extend Z to the whole interval $[0, N]$ by linearly interpolating between integer times. See Figure 1.10 for an example. Then a labelled looptree is entirely characterised by the pair of paths (W, Z).

In order to consider scaling limits, it is useful to describe the label process in terms of the Łukasiewicz path. Observe that the labels on the vertices are obtained by summing the label increments on the edges of the looptree on the path from the root which always follows each cycle on its left. Then each integer $i \in \{0, \dots, N\}$ such that $\Delta W_i \geq 0$ encodes a cycle of the looptree, with length $\Delta W_i + 1$. Let us denote by $(\xi^i_1, \dots, \xi^i_{\Delta W_i + 1})$ the label increments along the edges of the cycle, in the order induced by following the contour of the looptree. Then the label process is given by the formula: for every $0 \leq j \leq N$,

$$Z_j := \sum_{i=0}^{j-1} \sum_{k=0}^{\Delta W_i + 1} \left(\xi^i_1 + \cdots + \xi^i_k\right) \mathbf{1}_{\{\inf_{[i+1,j]} W = W_{i+1} - k + 1\}}. \tag{1.13}$$

We point out that the terms with $k = 0$ and $k = \Delta W_i + 1$ actually give a null contribution.

Let us finally recall that the labelling is good when each bridge $(\xi^i_1, \dots, \xi^i_{\Delta W_i+1})$ belongs to $\mathbf{B}^{\geq -1}_{\Delta W_i+1}$ defined in (1.11). In this case, the increments of Z all lie in $\mathbf{Z}_{\geq -1}$. We shall use the following simple representation which provides a way to sample conditionally given W a good labelling of the looptree uniformly at random. Indeed, this amounts to sampling independently a bridge uniformly at random in $\mathbf{B}^{\geq -1}_{\Delta W_i+1}$ for every i.

Lemma 1.2. *Let $(\xi_k)_{k\geq 1}$ be i.i.d. copies of a random variable ξ with the centred geometric distribution given by $\mathbf{P}(\xi = i) = 2^{-i-2}$ for every $i \geq -1$. Then for every $k \geq 1$, the sequence $(\xi_1, \dots, \xi_k)$ under the conditional law $\mathbf{P}(\cdot \mid \xi_1 + \cdots + \xi_k = 0)$ has the uniform distribution on the set $\mathbf{B}^{\geq -1}_k$.*

The vertex-counting process Recall the list $(c_0, \dots, c_N)$ of vertices of the looptree when following its contour. Since some are visited more than once, it contains redundancies which we want to remove in order to control the uniform distribution on the vertices. For every $t \in [0, N]$, let us therefore denote by $\Lambda(t)$ the number of vertices among $c_0, \dots, c_{\lfloor t \rfloor}$ which are fully visited, i.e. have all their external corners in this list. Then

$$\Lambda(t) = \#\{0 \leq i \leq \lfloor t \rfloor : \Delta W_i = -1\}.$$

Let $V_N = \Lambda(N)$ denote the total number of vertices of the looptree. Conversely for $t \in [0, V_N]$, let $\lambda(t) \in \{0, \dots, N-1\}$ be the index such that the vertex $c_{\lambda(\lceil t \rceil)}$ is the $\lceil t \rceil$'th vertex fully visited in the contour of the looptree. Note that the sequence $(c_{\lambda(0)}, \dots, c_{\lambda(V_N-1)})$ now lists the vertices *without redundancies*. In our random model, the following convergence in probability

$$\left(\frac{\Lambda(Nt)}{V_N}, \frac{\lambda(V_N t)}{N}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{\mathbf{P}} (t, t)_{t\in[0,1]} \tag{1.14}$$

holds for the uniform topology [Mar22a, Lemma 2.2]. Indeed by the previous formula, this is basically a weak law of large numbers for sampling without replacement in an urn. Under this assumption, sampling uniformly at random in the looptree a corner (which can be done simply by sampling an instant in $[0, N]$) or a vertex is asymptotically equivalent.

1.2.3 Tightness results

Recall that Step III towards the convergence of maps consists in proving that the label process associated with our random maps with prescribed face degrees is always tight when suitably normalised. Let us describe the idea of the proof which should explain how the coding by looptrees and thus Łukasiewicz paths simplifies the argument. To simplify the exposition, let us consider maps without boundary and let $M^{\mathbf{f}_n}$ denote a uniformly random map with n faces with degrees given by $\mathbf{f}_n$. Let LT^n denote the corresponding random looptree, with cycle lengths $\mathbf{f}_n$, equipped with a uniformly random labelling. Finally let (W^n, Z^n) be the corresponding coding paths. Note that we only indicate the dependence in n and not in $\mathbf{f}_n$ to lighten the notation.

Looptrees are tight As a first step we show that looptrees are always tight when rescaled by a factor σ_n [Mar22b, Proposition 5.1]. This was implicit in [Mar22a]. Let us abuse notation and for any $t \in [0, 1]$, let us call the 'vertex t' the vertex visited at time $\lfloor E_n t \rfloor$ in the contour sequence, where we recall that E_n is the number of edges. The idea is to rely on Kolmogorov's criterion and bound the moments of the distance between the vertices s and t for fixed $s, t \in [0, 1]$. Then take $r \in [s, t]$ to be the first instant such that any path from the vertex t to the root 0 must pass by r and let us bound the distance between s and r, i.e. the 'left branch'. In terms of the tree (visiting the vertices in the depth-first search order), r would the child of the last common ancestor of s and t which is an ancestor of t.

For this left branch, the idea is to upper the looptree distances in which one follows each loop between s and r by taking their shortest side between left and right, by simply the sum of the length of

their right part. The point is that in terms of the Łukasiewicz path, the total length of this right part equals

$$W^n_{\lfloor E_n s\rfloor} - \inf_{u\in[s,t]} W^n_{\lfloor E_n u\rfloor},$$

where our r is actually the instant that realises the infimum. Then we can use a Chernov bound for martingales with bounded negative increments to obtain an exponential decay of the negative tail of W^n, and so on the looptree distance from s to r, from which we easily derive a moment bound. Details can be found in [Mar22a, Section 4.2].

The right branch from r to t then uses a symmetry argument. It should be noted that the latter is not trivial since the construction of the looptree is not symmetric.

Labels are tight Tightness of the rescaled label process $(\sigma_n^{-1/2} Z^n_{E_n t}; t \in [0,1])$ is proved in [Mar22a, Section 5.1], the idea is close to the preceding one, namely fix $s < t$ and consider the difference of label between the vertices s and t. Then let us write this as the sum of the label increments along each cycle from s to t, the point is that, conditionally given the looptree, these increments are independent and, thanks to Lemma 1.2, they can be seen as bridges of a centred geometric random walk evaluated at certain times (given by the looptree, this is simply Formula (1.13)). Then it is not difficult to upper bound the conditional $2p$'th moment of $|Z^n_{E_n t} - Z^n_{E_n s}|$ by the p'th moment of the looptree distance between s and t. Tightness follows by further averaging with respect to the looptree using the previous result.

Maps are tight Finally, since the coding pair $(\sigma_n^{-1} W^n, \sigma_n^{-1/2} Z^n)$ is tight, then from every sequence of integers, one can extract a subsequence along which it converges. Then the argument briefly recalled in Section 1.1.4 proves that one can extract a further sub-subsequence along which $\sigma_n^{-1/2} M^{\mathfrak{f}_n}$ converges in the GH topology, and more precisely the distance function converges for the uniform topology to a random pseudo-distance D_∞. Actually the convergence holds in the GHP topology thanks to (1.14) which allows to eliminate redundancies in the list of vertices (recall Remark 1.1).

1.2.4 Convergence results

Let us finish with Step iv, that is, under the assumptions of Theorem 1.5 or Theorem 1.6 on the face degrees, prove that the Łukasiewicz path and label process converge in distribution. Recall that we consider here maps without boundary just to simplify the exposition.

Processes with exchangeable increments

The first point is that the convergence of the Łukasiewicz paths is already known in the literature. Precisely, recall the *Vervaat transform* which relates both in discrete (where it is also called conjugation operation or cyclic shift) and continuous settings an excursion path and a bridge by cyclically shifting the latter at its infimum. In the case of W^n, the random bridge Y^n is simply such that its increments form a random permutation of the $(\mathfrak{f}_{n,k} - 1)$'s, adding also $E_n - n$ null degrees, that is $\mathfrak{f}_{n,k} = 0$ for $E_n < k \le n$. By construction the joint law of these increments is invariant under permutation. In continuous time, processes with *exchangeable increments* are completely characterised and so are invariance principles. In our context, noting by simple manipulations that $\sigma_n^2 = \sum_{i=1}^n \mathfrak{f}_{n,i}(\mathfrak{f}_{n,i} - 1) = \sum_{i=0}^{E_n} (\Delta Y^n_i)^2 - 1$, we can rewrite [Kal02, Theorem 16.23] as follows.

Theorem 1.7 ([Kal02]). *The rescaled processes $(\sigma_n^{-1} Y^n_{E_n t})_{t\in[0,1]}$ converge in distribution in the Skorokhod topology to some limit, say Y, if and only if there exist $\theta_0 \ge 0$ and $\theta_1 \ge \theta_2 \ge \cdots \ge 0$ with $\sum_{i\ge1} \theta_i^2 = 1$ such that, as $n \to \infty$,*

$$\sigma_n^{-1} \mathfrak{f}_{n,i} \longrightarrow \theta_i \quad \textit{for every} \quad i \ge 1.$$

Moreover in this case the limit Y takes the form:

$$Y_t = \theta_0 \mathbb{b}_t + \sum_{i\geq 1} \theta_i(\mathbf{1}_{\{U_i\leq t\}} - t), \tag{1.15}$$

where $\mathbb{b}$ is a standard Brownian bridge and independently the U_i's are i.i.d. uniformly distributed on $[0,1]$.

We then transport this convergence to the Łukasiewicz paths using the Vervaat transform: by exchanging the pre- and post-supremum of both Y^n and Y, under the previous assumption, we get

$$\left(\sigma_n^{-1} W^n_{E_n t}; t\in[0,1]\right) \xrightarrow[n\to\infty]{(d)} (X_t; t\in[0,1]), \tag{1.16}$$

where X is the Vervaat transform of Y in (1.15). Note that some care is needed when defining the Vervaat transform, especially in the finite variation regime since in this case the path Y makes a jump at its 'instant of infimum', see [Mar22b, Section 6] for details. Theorem 3 in [AHUB20a] provides a sufficient condition for X to be the excursion version of Y, in the sense that it is the weak limit as $\varepsilon \to 0$ of the process Y conditioned on the event $\{\inf Y > -\varepsilon\}$. We do not consider this interesting question in general and only content ourselves with this construction of X.

Continuum labelled looptrees

Looptrees coded by a càdlàg path Curien & Kortchemski [CK14] defined a looptree from a càdlàg path X with no negative jump by mimicking the discrete construction with a Łukasiewicz path. In words, every positive jump, say $\Delta X_s > 0$ codes a cycle with length ΔX_s, whose points are the first hitting times after s of each level $(X_s - x; x\in[0,\Delta X_s])$, with the times corresponding to $x = 0$ (that is s) and $x = \Delta X_s$ being identified. However the paths X considered there all satisfied the following property:

$$X_t = \sum_{s\leq t} \max\Big(0, \inf_{[s,t]} X - X_{s-}\Big), \tag{1.17}$$

that is to say that the running supremum of the time- and space-reversed path, namely the function $s\mapsto \overline{X}^t_s = \sup_{r\in[0,s]}(X_{t-} - X_{(t-r)-})$ is pure jump. In the extreme case where X is continuous, the looptrees of [CK14] are thus reduced to a single point. In [Mar22b] we extended the construction to the general setting. Basically, in (1.17), the left-hand side is always larger than or equal to the right-hand side, and precisely, letting Leb stand for the Lebesgue measure, the difference equals

$$C_t = \mathrm{Leb}\Big(\Big\{\inf_{r\in[s,t]} X_r; s\in[0,t]\Big\}\Big). \tag{1.18}$$

This is a continuous function so it encodes a real tree $\mathcal{T}_C$ as in (1.1). We then add the tree distance to the looptree distance defined in [CK14]. This allows to interpolate between the 'pure jump' looptrees (when X satisfies (1.17)) and real trees (when X is continuous, so $X = C$). In general, the tree $\mathcal{T}_C$ should not be thought of the continuum tree for which X would play the role of the Łukasiewicz path; indeed, it is null in the setting of stable Lévy processes in [CK14], since the latter satisfies (1.17), whereas a nontrivial continuum stable Lévy tree may be constructed [DLG02, Duq03]. However for Lévy processes with no negative jump and Gaussian parameter $\beta > 0$, the path C coincides with β times the height process of [DLG02].

Random labels on continuum looptrees Recall from Section 1.1.3 that the Brownian sphere is constructed from the (head of the) Brownian snake $Z^{\mathbb{e}}$ driven by the Brownian excursion $\mathbb{e}$, which describes a Gaussian field on the continuum tree $\mathcal{T}_{\mathbb{e}}$. In the case of a looptree, although without this formalism (and this was one motivation to define looptrees in [CK14]), Le Gall & Miermont [LGM11] constructed a process that describes a Gaussian field on the looptree coded by X under the assumption (1.17). In this case, one places on each cycle of the looptree an independent Brownian bridge, with duration given

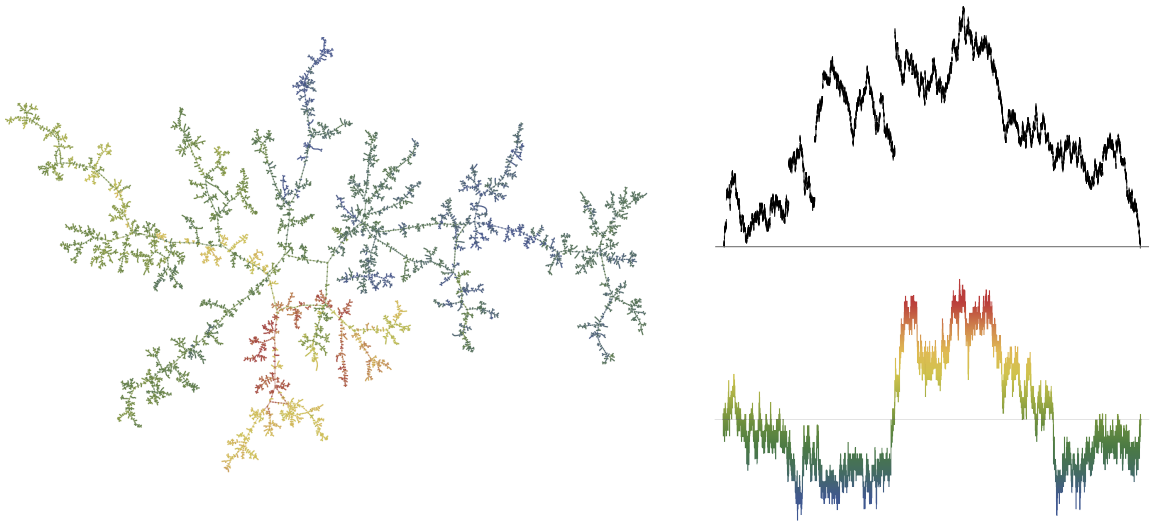


Figure 1.11 – *Left:* A large looptree with labels indicated by colours drawn non-isometrically in the plane. *Right:* Its Łukasiewicz path which converges as in (1.16) to a limit with $\theta_0 > 0$ and $\sum_i \theta_i = \infty$ on top and the label process on the bottom.

by the length of the cycle (which can be obtained by the usual diffusive scaling), which describes the increments of the field along the cycle. The value of the field at a given point is then the sum of the increments along a geodesic path to the root, analogously to (1.13) in the discrete setting, namely for all $t \in [0,1]$, enumerating the jump times of X as $(t_i)_{i\geq 1}$, they defined the label process at time t by:

$$\sum_{t_i<t} \sqrt{\Delta X_{t_i}}\, \mathbb{b}_i \left(\max\left(0, \frac{\inf_{[t_i,t]} X - X_{t_i-}}{\Delta X_{t_i}}\right)\right), \tag{1.19}$$

where the $\mathbb{b}_i$'s are i.i.d. standard Brownian bridges independent of X. Again, let us refer to [LGM11] for details on this construction, as well as to [Mar22b] with the formalism of looptrees. Without the assumption (1.17), when C defined in (1.18) is nontrivial, one then adds to (1.19) an independent Brownian snake Z^C driven by C, defined exactly as in Section 1.1.3. See Figure 1.11 for a simulation of a continuum labelled looptree.

Convergence of labelled looptrees

We now aim at showing the convergence of the rescaled label process, jointly with that of the Łukasiewicz path discussed above. Recall that we already know that the sequence is tight, so it remains to prove the convergence of the finite-dimensional marginals, which we may take as i.i.d. uniform random times independent of the rest. Let us only discuss the case of a single random time, the general case is not much different. The key technical result to deal with the continuous part Z^C of the limit is a so-called *spinal decomposition* in the looptree that describes the loops on a geodesic path from the root to a uniform random point, derived in [Mar22a].

A spinal decomposition In terms of the tree associated with the looptree, which is a more familiar setting, we are given a random vertex U^n and we consider the sequence $(K_i^n, J_i^n)_{1\leq i\leq |U^n|}$ where K_i^n is the number of children of the ancestor of U^n at height $i-1$ and $J_i^n \in \{0,\dots,K_i^n - 1\}$ is the number among these children that lie to the right of the next ancestor of U^n. Then straightforward but long calculations based on exact enumeration formulae allow to compare the law of this sequence with that of $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s given as follows:

(i) The $\mathcal{K}_i^n$'s are sampled without replacement from the size-biased empirical offspring distribution, so for each k, $i \geq 1$, we have $\mathbf{P}(\mathcal{K}_i^n = k) = E_n^{-1}\#\{i : \mathrm{f}_{n,i} = k\}$;

(ii) Conditionally given $\mathcal{K}_i^n$, the variable $\mathcal{J}_i^n$ has the uniform distribution on $\{0, \dots, \mathcal{K}_i^n - 1\}$.

By 'comparing' we mean more formally controlling the Radon–Nikodym derivative when replacing the true (K_i^n, J_i^n)'s by the $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s. Let us note that Broutin & Marckert [BM14] derived such a comparison in which the $\mathcal{K}_i^n$'s are i.i.d. sampled with replacement, but this leads to some difficulties outside the regime they consider, in which especially σ_n^2 is of order n. Also, in order to control all finite dimensional marginals, we actually need to consider simultaneously the ancestral lines of several marked vertices.

The case of no large faces Let us first focus on the Brownian sphere regime $\mathrm{f}_{n,1} \ll \sigma_n$ and discuss the large face regime after. We shall need to control the height $|U^n|$ in the tree of our uniform random vertex, let us see U^n as a uniform random time, then the Łukasiewicz path evaluated at this time relates to the previous (K_i^n, J_i^n)'s by

$$W_{U_n}^n = \sum_{i=0}^{|U_n|} J_i^n.$$

Let us replace the true values (K_i^n, J_i^n)'s by the preceding $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s, then the average growth of W^n at each generation becomes

$$\mathbf{E}\left[\mathcal{J}_i^n\right] = \mathbf{E}\left[\frac{\mathcal{K}_i^n - 1}{2}\right] = \frac{\sigma_n^2}{2E_n}.$$

By (1.16), if $\mathbb{e}$ is the Brownian excursion and U an independent random time, then $\sigma_n^{-1} W_{U_n}^n \longrightarrow \mathbb{e}_U$ in distribution. This suggests that

$$\frac{\sigma_n}{2E_n} \cdot |U_n| \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbb{e}_U. \tag{1.20}$$

This is indeed the case, and more generally the tree reduced to finitely many vertices converges to the reduced Brownian tree under the sole assumption $\mathrm{f}_{n,1} \ll \sigma_n$ [Mar22a, Theorem 2.4].

From the point of view of the looptree, the J_i^n's and $K_i^n - J_i^n$'s denote respectively the length of the right and left part of the loops on a geodesic from the random corner U_n to the root so this geodesic has length

$$d_{LT^n}(0, U_n) = \sum_{i=1}^{|U_n|} \min\{J_i^n, K_i^n - J_i^n\}.$$

Let us again replace these true values (K_i^n, J_i^n)'s by the preceding $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s, then at least in average, $\min\{\mathcal{J}_i^n, \mathcal{K}_i^n - \mathcal{J}_i^n\}$ is about $\mathcal{J}_i^n/2$, so the right-hand side above is roughly $\frac{1}{2} W_{U_n}^n$. As previously, this suggests that

$$\frac{2}{\sigma_n} d_{LT^n}(0, U_n) \quad \xrightarrow[n\to\infty]{(d)} \quad \mathbb{e}_U.$$

Actually boundary effects can occur depending on wether a cycle has odd or even length, leading to a possibly different constant. Nevertheless, it can be proved that $\sigma_n^{-1} LT^n$ converges to a multiple of the Brownian CRT when $\mathrm{f}_{n,1} \ll \sigma_n$, see [Mar22b, Section 7.2].

Finally let us consider the label of U_n. Recall from Lemma 1.2 that conditionally on the looptree, it is given by the sum of independent random bridges of a centred geometric random walk with parameter 1/2, conditioned to be back at 0 at the time K_i^n and evaluated at time J_i^n. Such a law is centred and Marckert & Miermont [MM07] have calculated the variance. Again replacing the (K_i^n, J_i^n)'s by the $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s, the unconditioned variance of a label increment along a loop on the spine after this replacement equals

$$\mathbf{E}\left[\frac{\mathcal{K}_i^n - 1}{3}\right] = \frac{\sigma_n^2}{3E_n}.$$

Since the label $Z_{U_n}^n$ is the sum of $|U_n|$ independent random variables which are centred and with such a variance, then, with some good control on the third moment, a CLT for independent but not identically

distributed random variables yields the convergence in distribution

$$\left(|U_n|\frac{\sigma_n^2}{3E_n}\right)^{-1/2} Z^n_{U_n} \xrightarrow[n\to\infty]{(d)} G$$

where G has the standard Gaussian law. Now by (1.20) we have

$$\frac{3}{2\sigma_n}|U_n|\frac{\sigma_n^2}{3E_n} = \frac{\sigma_n}{2E_n}\cdot|U_n| \xrightarrow[n\to\infty]{(d)} \mathbb{e}_U.$$

Recall that $Z^{\mathbb{e}}$ is the head of the Brownian snake driven by the Brownian excursion $\mathbb{e}$, i.e. $Z^{\mathbb{e}}_U$ has conditionally given $\mathbb{e}$ and U the Gaussian law with variance $\mathbb{e}_U$. We conclude from the last two displays that

$$\sqrt{\frac{3}{2\sigma_n}}\cdot Z^n_{U_n} \xrightarrow[n\to\infty]{(d)} Z^{\mathbb{e}}_U.$$

This proves the convergence to the head of the Brownian snake when no face has degree of order σ_n. The rerooting trick then finishes the proof of the convergence of the corresponding maps to the Brownian sphere (or disk if we were considering maps with a boundary).

The case of 'only large faces' Suppose now that in our discrete models we do have faces of degree of order σ_n. Assume also for the moment that the limit X of the Łukasiewicz paths satisfies the pure jump property (1.17). Then now the convergence of the looptree distance is deterministic: in the limit this distance can be approximated by the contribution of the (finitely many) loops longer than δ when $\delta \downarrow 0$. Now for each $\delta > 0$ fixed, these long loops are the limits of the loops longer than $\delta\sigma_n$, i.e. increments of W^n which converge to the large jumps of X. Then the corresponding values of (K_i^n, J_i^n) are coded by W^n and converge to the analogous quantities in X. Since the rest can be made arbitrarily small by taking δ small enough, we can pass to the limit. This argument was already used in the proof of Theorem 4.1 in [CK14] where another assumption was needed there on the height of the tree, but it is not needed here; recall indeed from Figure 1.6 that we do not consider exactly the same discrete models.

As for the labels, the same argument applies: we only consider the long loops, which correspond to large jumps of the Łukasiewicz paths, then the label increments along these long loops are roughly speaking independent random walk bridges of length given by the size of the increment and evaluated at certain times given in terms of W^n and they converge after a diffusive scaling to the Brownian bridges that we see in (1.19). Again if X satisfies (1.17) then the total contribution of the short loops vanishes in the limit when the length of the loops tends to 0. This argument can already be found in the proof of Proposition 7 in [LGM11].

The general case In general when X does not satisfy the pure jump property (1.17), the two effects mix. Roughly speaking, on the one hand the loops longer than $\delta > 0$ in the limit can be treated just as above, the difference is that the rest does not vanish. However now in this rest, all the loops are small, i.e. we only consider degrees smaller than $\delta\sigma_n$ in the discrete models, and this resembles the first regime $\mathrm{f}_{n,1} \ll \sigma_n$, except that now we work with δ fixed, let $n \to \infty$ and then let $\delta \to 0$. The argument sketched above in the regime with no large face can be extended with more work, with the process C from (1.18) instead of the Brownian excursion $\mathbb{e}$.

This was done in [Mar22b, Section 7]. We also proposed in Section 5.3 there a (consequently not easy to read at first glance) unified treatment of the idea of 'replacing the (K_i^n, J_i^n)'s by the $(\mathcal{K}_i^n, \mathcal{J}_i^n)$'s and averaging a function $g(k, j)$ along the spine' that, when applied to well-chosen functions g, easily yields the aforementioned results on the labels and tree and looptree distances by relating these quantities to the Łukasiewicz path and its scaling limit.

1.2.5 Comments and perspectives

Let us end this section by some perspectives to continue this work. Recall from Section 1.1.2 the two questions of studying random planar *graphs*, as well as maps canonically embedded in the sphere and LQG surfaces.

A stick-breaking point of view on (loop)trees Parallel to our work, Blanc-Renaudie [BR20, BR21, BR22] studied with success the same discrete and continuum objects, but with a completely different point of view. Indeed, no Łukasiewicz path nor exchangeable increment processes is used there, instead he relies on the idea of constructing (loop)trees by a stick-breaking procedure. This method is very powerful and allows him to obtain some results that we miss, such as characterising compactness of the continuum trees, providing a tightness criterion for discrete trees and thus their convergence in the GHP topology, or computing the fractal dimensions of the continuum (loop)trees.

We believe that both points of view help to understand the models. One direction of future research would be to better understand processes with exchangeable increments and Inhomogeneous Continuum Random Trees and related labelled looptrees and maps, and be able to pass from one representation to the other directly in the continuum setting. In particular extending the theory of Lévy trees [DLG02] in this context would be nice, but it does not seem easy as one would first need to develop an appropriate local time theory for exchangeable increment processes. A simple example of annoying points: we believe that, analogously to Lévy processes, the 'pure jump' condition (1.17) is equivalent to $\theta_0 = 0$, but we are not able to prove it in full generality (only when $\sum_i \theta_i < \infty$). For discrete trees, our method allows to prove the convergence of reduced subtrees, spanned by finitely many random vertices, but tightness is missing; these trees have attracted quite a lot of attention recently [Lei19, AHUB20b, ABBHK21, ABDMM21, BOHT21, ABD22].

More on random maps Of course the question of uniqueness of the subsequential limits is central. For stable maps (see next section), this is under active investigation by Curien, Miermont, & Riera and in the case $\theta_0 = 0$ the general strategy could probably be used, although extending their technical estimates to exchangeable increment processes promises technical difficulties. The case $\theta_0 > 0$ is another story since, as above one would need to mix the effects of the Gaussian part and the jumps. In another direction, the Hausdorff dimension of the subsequential limits of quadrangulations [LG07, Bet10, Bet15] or of maps with large random faces [LGM11] were computed before solving the uniqueness problem. As we mentioned Blanc-Renaudie [BR22] calculates the fractal dimensions of the continuum looptrees. We would like to be able to get these results directly from our construction. Also we expect the limit maps in Theorem 1.6 to have fractal dimensions which are just half those of the associated looptree.

Finally we must mention the question of non-bipartite maps. The most notable invariance principle in this case is due to Addario-Berry & Albenque [ABA21] (the case of the uniform distribution [BJM14] uses a very nice idea, but specific to this model which relates it to quadrangulations) who proved the convergence of p-angulations for any $p \geq 5$ odd. Recall that the correspondence from [BDFG04] applies to any plane maps, but it leads to a multitype labelled tree (three or four types depending on the convention). The convergence of random labelled trees to the Brownian snake on the Brownian CRT was proved by Miermont [Mie08a], but the labelled trees associated with p-angulations do not exactly fit in this framework, an issue caused by some *asymmetry* of the labels. This was discussed by Miermont [Mie06] who proved the convergence of the radius and profile of the maps. In [ABA21] the authors push further this reasoning and transfer the results of [Mie08a] to the asymmetric labelled trees. Now one can again rewrite the correspondence from [BDFG04] using looptrees, which have now two types of vertices (to take into account the edges of the maps whose both endpoints lie at the same distance from the distinguished vertex) and we hope that this point of view can remove some technicalities in order to generalise the results of [ABA21].

1.3 Boltzmann random maps

We describe in this section the papers [Mar18a, Mar22b, KM21] on the model of random (bipartite) Boltzmann maps, the last one in collaboration with Igor Kortchemski. We also briefly mention the paper [Mar20] and its differences with the other ones.

1.3.1 Boltzmann maps and Bienaymé–Galton–Watson (loop)trees

Marckert & Miermont [MM07] introduced a model of random plane maps designed to have random face degrees, called *Boltzmann distributions*. They are also related to statistical physics model on maps, especially with the $O(N)$ loop model. In a few words, one aims at understanding the behaviour of random maps, say triangulations or quadrangulations, sampled together with loops on them or one their dual (one may think of interfaces of percolation or Ising clusters for example). A simplified version consists in emptying the regions surrounded by a loop (the so-called 'gasket decomposition'), and this gives a Boltzmann map; in a second step, one then wants to recover the full model by gluing in each hole a Boltzmann map with the corresponding boundary, and so on. See [LGM11, Section 8] as well as [BBG12, CCM20] for more details.

As in the previous section, in order to simplify the exposition and lighter the notation, we only consider here maps without boundary (Boltzmann maps with a boundary were introduced in [BM17]). In this model, we are given as a parameter a sequence $\mathbf{q} = (q_k)_{k\geq 0}$ of nonnegative real numbers, and we define a measure on the set of bipartite finite maps $\mathbf{M}$ by

$$\mathbf{Q}^{\mathbf{q}}(M) = \prod_{f\in F(M)} q_{\deg(f)/2} \qquad \text{for } M \in \mathbf{M}.$$

Define also a measure on pointed maps $(M, v_\star) \in \mathbf{M}_\star$ by setting $\mathbf{Q}^{\mathbf{q}}_\star((M, v_\star)) = \mathbf{Q}^{\mathbf{q}}(M)$. Relying on the bijection from [BDFG04] between pointed maps and labelled mobiles, Marckert & Miermont [MM07] provided an analytical condition to ensure that $\mathbf{Q}^{\mathbf{q}}_\star$ has a finite total mass, in which case $\mathbf{Q}^{\mathbf{q}}$ also has finite mass and thus both can be rescaled it into probability distributions:

$$\mathbf{P}^{\mathbf{q}}_\star(\cdot) = \frac{\mathbf{Q}^{\mathbf{q}}_\star(\cdot)}{\mathbf{Q}^{\mathbf{q}}_\star(\mathbf{M}_\star)} \qquad \text{and} \qquad \mathbf{P}^{\mathbf{q}}(\cdot) = \frac{\mathbf{Q}^{\mathbf{q}}(\cdot)}{\mathbf{Q}^{\mathbf{q}}(\mathbf{M})}.$$

A (pointed) map sampled from $\mathbf{P}^{\mathbf{q}}$ or $\mathbf{P}^{\mathbf{q}}_\star$ is called a $\mathbf{q}$-*Boltzmann map*. Later, Bernardi, Curien, & Miermont [BCM19] proved that the measure $\mathbf{Q}^{\mathbf{q}}_\star$ on pointed maps has finite mass *if and only if* $\mathbf{Q}^{\mathbf{q}}$ on non pointed maps does. This is summarised in the next lemma where we use the notation

$$\overline{q}_0 = 1 \qquad \text{and} \qquad \overline{q}_k = \binom{2k-1}{k-1} q_k \quad \text{for} \quad k \geq 1.$$

Lemma 1.3 ([BCM19, MM07]). *The following three conditions are equivalent:*

(i) The measure $\mathbf{Q}^{\mathbf{q}}$ has a finite total mass.

(ii) The measure $\mathbf{Q}^{\mathbf{q}}_\star$ has a finite total mass.

(iii) The generating series $g_{\mathbf{q}} : x \mapsto \sum_{k\geq 0} x^k \overline{q}_k$ has at least one fixed point.

Note that $g_{\mathbf{q}}$ is continuous, convex, strictly increasing, and satisfies $g_{\mathbf{q}}(0) = 1$ so its has at most two fix points and the smallest one, if any, is $x_{\mathbf{q}} > 1$ and satisfies $g'_{\mathbf{q}}(x_{\mathbf{q}}) \leq 1$.

Definition 1.2. We say that $\mathbf{q}$ is *admissible* when $g_{\mathbf{q}}$ in Lemma 1.3 has at least one fixed point. In this case, if $x_{\mathbf{q}} > 1$ denotes the smallest fixed point, then the sequence given for all $k \geq 0$ by

$$\mu_{\mathbf{q}}(k) = x_{\mathbf{q}}^{k-1}\overline{q}_k = x_{\mathbf{q}}^{k-1}\binom{2k-1}{k-1} q_k \tag{1.21}$$

defines a probability measure with mean $g'_{\mathbf{q}}(x_{\mathbf{q}}) \leq 1$. We say that $\mathbf{q}$ is *critical* when $g'_{\mathbf{q}}(x_{\mathbf{q}}) = 1$, otherwise it is *subcritical*.

The correspondence from Lemma 1.1 transports the measure $\mathbf{Q}^{\mathbf{q}}_\star$ on the set of finite well labelled looptrees (LT, ℓ), where the weights are now on the cycle lengths. In terms of trees which are more familiar, i.e. as in Figure 1.9, this provides a measure on labelled trees (T, ℓ), where the weights are on the offspring numbers of the internal vertices. Recall the set of bridges $\mathbf{B}_k^{\geq -1}$ from (1.11) that a good labelling induces on each cycle with length $k \geq 1$ and notice that its cardinal is $\#\mathbf{B}_k^{\geq -1} = \binom{2k-1}{k-1}$. Then $\mathbf{Q}^{\mathbf{q}}_\star$ induces a measure on unlabelled finite trees T given precisely by

$$\mathbf{Q}^{\mathbf{q}}(T) = \prod_{u \in V(T)} \overline{q}_{k_u},$$

where k_u is the offspring number of the vertex u. Random trees sampled proportionally to such a weight are called *simply generated* [Jan12]. In the case when $\mathbf{q}$ is a probability distribution, these are known as *Bienaymé–Galton–Watson* trees.

It is easy to check that if $\mathbf{q}$ is admissible, then the rescaled laws $\mathbf{P}^{\mathbf{q}}_\star$ and $\mathbf{P}^{\mu_{\mathbf{q}}}_\star$ coincide. Therefore the labelled tree associated with a random pointed map sampled from $\mathbf{P}^{\mathbf{q}}_\star$ has the following distribution:

(i) First the tree has the same law as a Bienaymé–Galton–Watson with subcritical or critical offspring distribution $\mu_{\mathbf{q}}$ as in (1.21), abbreviated $\mu_{\mathbf{q}}$-BGW tree.

(ii) Second conditionally on the tree, the good labelling is sampled uniformly at random, which can be done by sampling independently for each internal vertex a geometric random walk bridge as in Lemma 1.2.

Finally recall that the Łukasiewicz path of a $\mu_{\mathbf{q}}$-BGW tree has the law of a random walk with step distribution $\mu_{\mathbf{q}}(\cdot + 1)$ on $\mathbf{Z}_{\geq -1}$ killed when first hitting -1. Note that the sequence $\mathbf{q}$ is critical when the offspring distribution is critical, i.e. when it has mean 1; equivalently this corresponds to requiring that the Łukasiewicz walk is centred. A key point is the following remark, which was used on the unlabelled tree side by Broutin & Marckert [BM14] to recover in this way Aldous' pioneer result on the convergence of size-conditioned BGW trees [Ald93].

Remark 1.4. Since the Boltzmann weights only depend on the face degrees, then Boltzmann maps can be seen as mixtures of the model studied in Section 1.2 with prescribed face degrees, in which these degrees (and the number of faces itself) are first sampled at random. Therefore we can deduce invariance principle for such maps by applying the general results of Section 1.2.1 to these random degrees. Note that the scaling factor σ_n^2 in (1.10) used there is now random.

1.3.2 Monoconditioned maps: the stable maps

Let us start with a simple application of the strategy in Remark 1.4 to recover and extend several known results, we then elaborate on this idea in Section 1.3.3 to study a new model. We aim at conditioning our $\mathbf{q}$-Boltzmann maps to be large. Recall that we consider maps without boundary to simplify the exposition. By Remark 1.4, this corresponds to conditioning the associated (loop)tree, or equivalently the Łukasiewicz path. Namely a pointed map has n vertices, n edges, or n faces respectively when the corresponding tree has $n - 1$ leaves, n edges, or n internal vertices respectively, and equivalently when its Łukasiewicz path has $n - 1$ negative increments, $n + 1$ total increments, or n nonnegative increments respectively. Recall that in any case, the Łukasiewicz path is a conditioned random walk with step distribution $\mu_{\mathbf{q}}(\cdot + 1)$ on $\mathbf{Z}_{\geq -1}$, where $\mu_{\mathbf{q}}$ is defined in (1.21) and is stopped at the fist hitting time of -1. We assume for the rest of this section that $\mu_{\mathbf{q}}$ satisfies the following assumption.

Definition 1.3. Fix $\alpha \in (1,2)$. A weight sequence $\mathbf{q}$ is said to be *discrete α-stable* when it is critical and such that there exists a constant $c_{\mathbf{q}} > 0$ such that

$$\mu_{\mathbf{q}}([k,\infty)) \underset{k\to\infty}{\sim} c_{\mathbf{q}} \cdot k^{-\alpha}. \tag{1.22}$$

It is said to be *discrete 2-stable* when it is critical and $\mu_{\mathbf{q}}$ has finite variance.

One could consider the general *domain of attraction of a stable law*, when the constant $c_{\mathbf{q}}$ in (1.22) is replaced by a *slowly varying function* (which allows also to consider the cases $\alpha = 1$ with finite mean and $\alpha = 2$ with infinite variance). This is done in [Mar18b, Mar22a, Mar22b] and our restriction here is made only for better readability. Recall that in this case, there exists another constant $C_{\mathbf{q}} > 0$ such that if $S = (S_n)_{n\geq 0}$ is a random walk with step distribution $\mu_{\mathbf{q}}(\cdot + 1)$, then we have

$$\left(C_{\mathbf{q}} n^{-1/\alpha} S_{\lfloor nt\rfloor}; t \geq 0\right) \xrightarrow[n\to\infty]{(d)} \left(Y^{\alpha}_t; t \geq 0\right), \tag{1.23}$$

for the Skorokhod topology, where Y^{α} is an α-stable Lévy process with no negative jump, characterised by $\mathbf{E}[\exp(-\lambda Y^{\alpha}_t)] = \exp(t\lambda^{\alpha})$ for every $t, \lambda \geq 0$. In particular $Y^2/\sqrt{2}$ is a standard Brownian motion.

Pointed maps and conditioned walks

Fix a critical discrete α-stable weight sequence $\mathbf{q}$ and consider pointed maps sampled from the conditional law $\mathbf{P}^{\mathbf{q}}_{\star}(\cdot \mid \text{size} = n)$, where 'size' denotes either the number of vertices, of edges, or of faces. By Remark 1.4, we aim at controlling the asymptotic behaviour of the random face degrees in order to apply the general results of Section 1.2.1; these degrees are twice the nonnegative increments plus one of the corresponding Łukasiewicz path W^n, which is a conditioned version of the walk S as above.

In the case $\alpha = 2$, when $\mu_{\mathbf{q}}$ has finite variance, we proved in [Mar18b] a weak law of large number for the square of the increments of W^n, showing that the random scaling σ_n^2 as defined in (1.10) is asymptotically equivalent to some deterministic constant times n. Moreover in this regime, the conditioned walk rescaled by a factor $\sqrt{n}$ converges to the Brownian excursion by [Kor12]. Since the Brownian excursion has no jump, then we fit in the case $\mathrm{f}_{n,1} \ll \sigma_n$ and we may conclude from Theorem 1.5 to the convergence of such pointed maps to the Brownian sphere, at a scaling of order $n^{1/4}$. This was extended to the full domain of attraction of a Gaussian law in [Mar22a], thus recovering this result first proved in [Mar18a].

The work [Mar18a] considers also the case $\alpha \in (1,2)$ (again in the full domain of attraction of an α-stable law), following the pioneer work of Le Gall & Miermont [LGM11]. Here the rescaled Łukasiewicz path scales like $n^{1/\alpha}$ and precisely by [Kor12], it converges in law to an excursion version of Y^{α} which now possesses macroscopic jumps. Here again σ_n^2 behaves asymptotically like some deterministic constant times $n^{2/\alpha}$ so Theorem 1.4 shows an abstract tightness result of the maps, at the scale $n^{1/(2\alpha)}$. Theorem 1.6 further provides some results about the subsequential limits and the convergence without extraction of the profile of distances, which recovers [Mar18a]. Note that the θ_i's from Theorem 1.6 are here random, they are given by the ranked jump sizes of the α-stable Lévy bridge.

Let us mention that [Mar22a] also considers the more rare case $\alpha = 1$ of the Cauchy domain of attraction, relying on recent results by Berger [Ber19] and Kortchemski & Richier [KR19]. This case falls into the so-called 'condensation regime', with a giant face which carries all the contribution to σ_n^2, and which can be though of as the giant boundary in Theorem 1.5. In this regime, the rescaled Łukasiewicz path converges in some sense to the deterministic drift $(1-t)_{0\leq t\leq 1}$ and the rescaled map to the Brownian CRT at the scale $\sqrt{n}$ (times a slowly varying function). This is also the case when $\mathbf{q}$ is subcritical, i.e. when the Łukasiewicz path has a negative drift, as originally shown by Janson & Stefánsson [JS15].

Non pointed maps

One may also want to consider non pointed maps. Observe that if one first samples a pointed map from the conditional law $\mathbf{P}_\star^{\mathbf{q}}$ and then 'forgets the distinguished vertex', i.e. formally takes the projection on non pointed maps, then the resulting random map is biased by its number of vertices. Therefore, taking as notion of size of a map its number of vertices, the previous results apply to random non pointed maps sampled from $\mathbf{P}^{\mathbf{q}}(\cdot \mid n \text{ vertices})$, in which we further distinguish a vertex independently and uniformly at random. For the conditioning by the number of edges or of faces, one can easily compare the laws on pointed and non pointed maps and show that the bias disappears in the limit. Roughly speaking, one can show that these maps, when conditioned to have n edges, typically have about $v_{\mathbf{q}}n$ vertices for some $v_{\mathbf{q}} \in (0,1)$, and $(1-v_{\mathbf{q}})n$ faces, so each choice of size only modifies the results by a multiplicative constant. This trick is originally due to Bettinelli, Jacob, & Miermont [BJM14] and has since been used in several works [Abr16, Ste18, BM17, Mar18b, Mar18a, Mar22a].

1.3.3 Biconditioned maps

Recently with Igor Kortchemski [KM21] we studied these Boltzmann maps conditioned by their number of vertices, edges, and faces *at the same time* and specifically asked about their behaviour when these three quantities do not follow the asymptotic linear relations as above. Since Euler's formula relate them, there are actually only two degrees of freedom. Throughout this subsection, we are thus given for every $n \geq 1$ an integer $K_n \leq n$ and we denote by M_{n,K_n} a bipartite map with

$$n-1 \text{ edges}, \qquad K_n+1 \text{ vertices}, \qquad \text{and so} \qquad n-K_n \text{ faces}.$$

In order to discard degenerate cases, we assume that both K_n and $n-K_n$ tend to infinity.

In the case of uniformly chosen such maps (both in the bipartite or general case) Fusy & Guitter [FG14, Section 6] predicted that for $b, c \in (0,1)$, typical distances in maps with n edges and n^b faces are of order $n^{(2-b)/4}$, whereas distances in uniform random maps with n edges and n^c vertices are of order $n^{c/4}$. We confirm in Theorem 1.8 below this prediction in the bipartite case and prove furthermore that in both cases the limit is the Brownian sphere. Specifically, for $0 < x < 1$, set

$$r(x) = \frac{(1-x)(3+x+\sqrt{(1-x)(9-x)})}{12x}.$$

Note that r is continuous, decreasing, and $r(x) \sim 1/(2x)$ and $r(1-x) \sim x/3$ as $x \to 0$.

Theorem 1.8 ([KM21]). *Let $(K_n)_{n\geq 1}$ be integers such that $K_n \to \infty$ and $n-K_n \to \infty$ and let M_{n,K_n} be a bipartite map with $n-1$ edges and K_n+1 vertices sampled uniformly at random. Then the convergence in distribution*

$$\left(r\left(\frac{K_n}{n}\right)\frac{9}{4n}\right)^{1/4} M_{n,K_n} \xrightarrow[n\to\infty]{(d)} \mathcal{S}$$

holds for the Gromov–Hausdorff–Prokhorov topology, where $\mathcal{S}$ is the Brownian sphere.

Remark 1.5. (i) The aforementioned prediction of [FG14] concerns the cases $K_n = n^c = o(n)$ and $n-K_n = n^b = o(n)$ respectively, which follow from the behaviour of r near 0 and 1.

(ii) This result is consistent with the work of Abraham [Abr16] who proved that bipartite maps with n edges sampled uniformly random converge in distribution to the Brownian sphere once rescaled by $(2n)^{-1/4}$. Indeed as proved there, such a map typically has about $2n/3$ vertices and $r(2/3) = 2/9$.

This theorem is reminiscent of the work of Labarbe & Marckert [LM07], which shows that uniformly random plane trees with n edges and K_n leaves, under the same non-degeneracy assumption $K_n \to \infty$ and $n-K_n \to \infty$, always converge to the Brownian CRT coded by the standard Brownian excursion once suitably normalised. More precisely, the corresponding contour process converges to the Brownian

excursion. However their work is restricted to the uniform case, using clever decompositions and exact counting formulae, whereas Theorem 1.8 is actually a corollary of more general results on Boltzmann maps, applied to the weight sequence $q_k = 1$ for all $k \geq 1$. Let us refer to [KM21] for general statements.

We shall use the same strategy as previously, based on Remark 1.4, which allows to conclude to the convergence of the map by controlling the degrees, and thus only looking at the Łukasiewicz path of the associated (loop)tree, whereas [LM07] deals with the contour processes. Studying contour process of general biconditioned random trees is one of the perspectives discussed in Section 1.3.5. Ultimately, a key point leading to Theorem 1.8 is new local limit estimates for random walks, outside the usually studied regimes, which we believe to be of independent interest. Let us sketch the general method and main argument leading to results such as Theorem 1.8 to see how these estimates come into play. We shall not however provide precise statements and refer to [KM21] instead.

Retro-engineering

As for mono-conditioned maps, we may rely on Theorem 1.5 and only study the Łukasiewicz path W_{n,K_n} of the associated labelled (loop)tree, which is a random walk with step distribution $\mu_{\mathbf{q}}(\cdot + 1)$ on $\mathbf{Z}_{\geq -1}$, where $\mu_{\mathbf{q}}$ is defined by (1.21), conditioned both to first hit -1 for the first time at time n *and* by making its K_n'th negative increment. Precisely, $a_n^{-1/2} M_{n,K_n}$ converges to the Brownian sphere if we can prove that the random scaling σ_n^2 as defined in (1.10) is asymptotically equivalent to some deterministic constant times a_n^2 and further that the largest increment of W_{n,K_n} is much smaller than a_n.

In our regimes, the first point can be checked by looking at the moments, possibly truncated, as for standard laws of large numbers, and will not be discussed here. We prefer instead to focus on proving that once suitably rescaled, these biconditioned excursions have no macroscopic increment. We actually prove that under suitable assumptions on both the weight sequence $\mathbf{q}$ and the number of negative increments K_n, these rescaled paths converge in distribution to the Brownian excursion. Let us sketch the proof in reverse direction.

From biconditioned excursions to nondecreasing bridges Thanks to the Vervaat transform, it is equivalent to prove that random bridges B_{n,K_n}, i.e. random walks with step distribution $\mu_{\mathbf{q}}(\cdot + 1)$ conditioned both to lie at position -1 at time n *and* to have made K_n negative increments in total, once suitably rescaled, converge in distribution to the Brownian bridge. Here a first key idea is to split in B_{n,K_n} the contribution of the negative and nonnegative increments. Precisely, let us denote by $L_n(k) = \#\{i \leq k : \Delta B_{n,K_n} = -1\}$ for every $k \leq n-1$. Notice that the position of the K_n negative increments in such a bridge is a uniform random choice of K_n positions among n available. In particular, it is fairly easy to prove that

$$\left(\frac{L_n(\lfloor nt \rfloor) - K_n t}{\sqrt{K_n(n-K_n)/n}} ; t \in [0,1] \right) \xrightarrow[n\to\infty]{(d)} \mathbb{b},$$

where $\mathbb{b}$ is a Brownian bridge (this is a – very – particular case of Theorem 1.7). Recall that we assume that $n - K_n \to \infty$, then this implies that

$$\left(\frac{nt - L_n(\lfloor nt \rfloor)}{n - K_n} ; t \in [0,1] \right) \xrightarrow[n\to\infty]{\mathbf{P}} (t ; t \in [0,1]).$$

On the other hand define the path S^+_{n,K_n} by *removing* the negative increments of B_{n,K_n}, which starts at 0, is nondecreasing, and ends at the position $K_n - 1$ at time $n - K_n$. Then this path is independent of L_n. Moreover the original bridge B_{n,K_n} is given by the difference between this nondecreasing path, time-changed using L_n, and L_n itself, namely for every $0 \leq i \leq n$, we have

$$\begin{aligned} B_{n,K_n}(i) &= S^+_{n,K_n}(i - L_n(i)) - L_n(i) \\ &= \left(S^+_{n,K_n}(i - L_n(i)) - K_n \frac{i - L_n(i)}{n - K_n} \right) - \frac{n}{n-K_n} \left(L_n(i) - K_n \frac{i}{n} \right). \end{aligned}$$

Let us extend S^+_{n,K_n} by linear interpolation and suppose for a moment that K_n/n converges to some limit $\tau \in (0,1)$ and that, for some constant $c > 0$, we can prove the convergence of the fluctuations

$$\Big(\frac{S^+_{n,K_n}((n-K_n)t) - K_n t}{c\sqrt{n}} ; t \in [0,1] \Big) \xrightarrow[n\to\infty]{(d)} \mathbb{b}', \tag{1.24}$$

where $\mathbb{b}'$ is a Brownian bridge independent from $\mathbb{b}$. Then combining the last displays, we derive that

$$\Big(\frac{1}{\sqrt{n}} B_{n,K_n}(nt); t \in [0,1] \Big) \xrightarrow[n\to\infty]{(d)} \Big(c\,\mathbb{b}'_t - \sqrt{\frac{\tau}{1-\tau}}\,\mathbb{b}_t ; t \in [0,1] \Big) \overset{(d)}{=} \Big(\sqrt{c^2 + \frac{\tau}{1-\tau}}\,\mathbb{b}_t ; t \in [0,1] \Big).$$

We conclude via the Vervaat transform again that

$$\Big(\frac{1}{\sqrt{(c^2 + \frac{\tau}{1-\tau})n}} W_{n,K_n}(nt); t \in [0,1] \Big) \xrightarrow[n\to\infty]{(d)} \mathbb{e}$$

as soon as (1.24) holds and $K_n/n \longrightarrow \tau \in (0,1)$.

In fact this sketch can be adapted when K_n/n tends either to 0 or to 1, or when in (1.24) the scaling factor is not of order $\sqrt{n}$, or when the limit is not a Brownian bridge. Nevertheless the idea is the same and only requires more care in adding the two contributions, see [KM21, Section 4.4] for details.

Convergence of nondecreasing bridges from local estimates The previous sketch was based on the convergence (1.24). The general setting is as follows: we are given a nondecreasing integer-valued random walk $S = (S_k; k \geq 0)$ which is conditioned to end at a given time n (previously $n - K_n$) at a given position x_n (previously $K_n - 1$). We aim at controlling the deviation around the average drift with slope x_n/n and prove that it converges to a Brownian bridge as in (1.24) or to another process. Let us describe a simple setting in which this holds.

First note that it suffices to prove a convergence as in (1.24) on any time interval $[0,u]$ for any fixed $u \in (0,1)$; indeed since the conditioned path is invariant under time- and space-reversal, then the remaining part of the path is then automatically tight. The Markov property applied at time u results in an absolute continuity between the conditioned and the unconditioned walk, namely if we let $\varphi_i(k) = \mathbf{P}(S_i = k)$ for every $i \geq 0$ and $k \in \mathbf{Z}$, then for every continuous and bounded function $F : \mathbf{D}([0,u],\mathbf{R}) \longrightarrow \mathbf{R}$ and any scaling factor v_n, it holds

$$\mathbf{E}\left[F\Big(\Big(\frac{S_{\lfloor nt \rfloor} - x_n t}{v_n} ; t \leq u \Big) \Big) \,\Big|\, S_n = x_n \right] = \mathbf{E}\left[F\Big(\Big(\frac{S_{\lfloor nt \rfloor} - x_n t}{v_n} ; t \leq u \Big) \Big) \cdot \frac{\varphi_{n-\lfloor un \rfloor}(x_n - S_{\lfloor un \rfloor})}{\varphi_n(x_n)} \right]. \tag{1.25}$$

Now suppose that the walk is aperiodic, has a finite second moment, and that the endpoint is typical in the sense that:

$$|x_n - n\mathbf{E}[S_1]| = o(\sqrt{n}).$$

Then the Local Central Limit Theorem (hereafter abbreviated LLT) shows that if d_t is the density of a centred Gaussian random variable with variance t, then for $v_n = \sqrt{n\mathrm{Var}(S_1)}$ it holds

$$\sup_{k\geq 0} \Big| v_n \cdot \varphi_{\lfloor nt \rfloor}(k) - d_t\Big(\frac{k - x_n t}{v_n} \Big) \Big| \xrightarrow[n\to\infty]{} 0. \tag{1.26}$$

This directly implies the convergence in distribution of $v_n^{-1}(S_{\lfloor nt \rfloor} - x_n t)$ to the Brownian motion X_t for any $t > 0$ fixed, which, by e.g. [Kal02, Theorem 16.14], actually suffices to conclude to the convergence of the whole unconditioned path, namely

$$\Big(\frac{1}{v_n} \big(S_{\lfloor nt \rfloor} - x_n t \big) ; 0 \leq t \leq 1 \Big) \xrightarrow[n\to\infty]{(d)} (X_t; 0 \leq t \leq 1).$$

Combining (1.25), the LLT (1.26), and Skorokhod's representation theorem, we infer that for every continuous and bounded function $F : \mathbf{D}([0,u],\mathbf{R}) \longrightarrow \mathbf{R}$,

$$\mathbf{E}\left[F\left(\left(\frac{S_{\lfloor nt\rfloor} - x_n t}{v_n}; t \le u\right)\right) \,\middle|\, S_n = x_n\right] \underset{n\to\infty}{\longrightarrow} \mathbf{E}\left[F(X_t; 0 \le t \le u)\frac{d_{1-u}(-X_u)}{d_1(0)}\right].$$

Finally, as for discrete random walks, the right-hand side is the absolute continuity relation between the Brownian motion and the Brownian bridge [Ber96, Chapter VIII.3].

The previous example shows that the convergence to a Brownian bridge directly follows from the local estimate (1.26). Actually this extends to more general Lévy processes which admit regular densities d_t. Note that (1.26) used two ingredients: the fact that the walk S had a finite second moment and the fact that x_n was close enough to the typical value. Borovkov & Borovkov [BB08, Theorem 6.1.5] provide a generalisation in a 'bulk regime' (or 'Cramér zone'), in which case the scaling factor v_n is again of order $\sqrt{n}$. This is sufficient to prove (1.24) in this form and ultimately Theorem 1.8 when K_n/n stays bounded away from both 0 and 1. For the two other regimes, when $K_n/n \longrightarrow 0$ or $K_n/n \longrightarrow 1$, we prove in [KM21, Theorem 1.1] new Gaussian local limit estimates of the form (1.26) especially designed for the cases $x_n/n \longrightarrow 0$ and $x_n/n \longrightarrow \infty$. Let us mention that we also provide in [KM21, Theorem 2.2] local estimates when the walk belongs to the domain of attraction of a stable law with index $\alpha \in (1,2]$, with either Gaussian or non Gaussian limits, depending on K_n.

Convergence of biconditioned random maps

Let M_{n,K_n} be a $\mathbf{q}$-Boltzmann map conditioned to have $n-1$ edges and K_n+1 vertices. Extracting a subsequence if necessary, we may always assume that we are in one of the cases $K_n/n \longrightarrow 0$, $K_n/n \longrightarrow 1$, or $K_n/n \longrightarrow \tau \in (0,1)$. In each case we provide sufficient conditions for a Gaussian local estimate to hold for the associate random walk, which by the preceding argument yields via Theorem 1.5 the convergence to the Brownian sphere. It turns out that the uniform distribution on biconditioned maps satisfies the assumptions in each case, which provides the unified statement given in Theorem 1.8, which is thus simply a consequence of more general results.

We also mentioned possibly non Gaussian limits. For simplicity, suppose that $\mathbf{q}$ is discrete α-stable in the sense of Definition 1.3, with $1 < \alpha < 2$. Let us write K_n of the form

$$K_n = \mu_{\mathbf{q}}(0)n + \lambda_n.$$

If $n^{-1/\alpha}\lambda_n \longrightarrow -\infty$, then we recover the Brownian sphere in the limit, whereas if $n^{-1/\alpha}\lambda_n \longrightarrow \infty$, it gives the Brownian tree [KM21, Theorem 5.4]. On the other hand, if $n^{-1/\alpha}\lambda_n \longrightarrow \lambda \in \mathbf{R}$, then the Łukasiewicz path, rescaled by $n^{1/\alpha}$ converges to the normalised excursion of an α-stable Lévy process with a drift explicitly given in terms of λ. Then Theorem 1.4 leads an abstract tightness which is completed by Theorem 1.6. These subsequential limits form a two-parameter family (α,λ) of metric spaces. We believe that for any α fixed, they are absolutely continuous with respect to the stable maps of [LGM11] which correspond to $\lambda = 0$, and finally that letting $\lambda \longrightarrow \pm\infty$, they interpolate between the Brownian sphere and the Brownian tree. This is one of our motivations to develop a general theory of non-stable 'Lévy random maps' briefly mentioned at the end of this section.

1.3.4 *Intermezzo*: random snakes with stable branching

Recall the model of random labelled trees associated with quadrangulations from the CVS bijection: the tree has the uniform distribution on plane trees with $n+1$ vertices, and the label of the vertices are such that the root has label 0 and the increments along the edges are i.i.d. with the uniform distribution on $\{-1,0,1\}$. In [Mar20] we consider a more general model in which:

(i) We first sample a Bienaymé–Galton–Watson tree T^n with some offspring distribution μ and conditioned to have $n+1$ vertices in total.

(ii) Then independently of T^n, we consider a sequence of i.i.d. copies of a given random variable L indexed by the n edges of the tree, and we define the label of every vertex as the sum of these random variables along its ancestral line, the root having label 0.

Suppose henceforth that μ is critical and discrete stable with index $\alpha \in (1,2]$, in the restricted sense of Definition 1.3 again just to simplify the exposition and remove the slowly varying sequences. Recall the stable Lévy process with no negative jump Y^α that appears in (1.23). Let us denote by W^n and C^n the Łukasiewicz path and contour process of T^n respectively. In this regime, Duquesne [Duq03] has proved the convergence in distribution

$$\left(\frac{C_{\mathrm{q}}}{n^{1/\alpha}} W^n_{\lfloor nt\rfloor}, \frac{1}{C_{\mathrm{q}} n^{(\alpha-1)/\alpha}} C^n_{2nt}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (X^\alpha_t, H^\alpha_t)_{t\in[0,1]}, \tag{1.27}$$

where X^α is the normalised excursion of Y^α and H^α is the associated continuous *height process*. In the case $\alpha = 2$, the processes X^α and H^α are equal, both to $\sqrt{2}$ times the standard Brownian excursion.

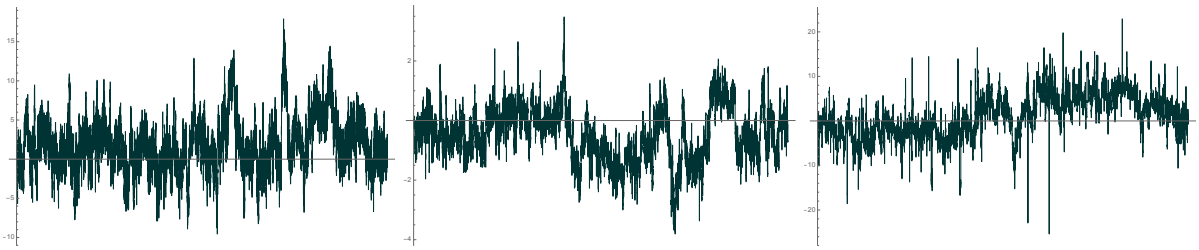
As in the previous section, let us code the labels by a continuous path $Z^n = (Z^n_{2nt}; t \in [0,1])$, following the contour of the tree. The most general results on scaling limits of Z^n are due to Janson & Marckert [JM05] who considered the case where μ has finite exponential moments, i.e. satisfies $\sum_{k\geq 0} \mathrm{e}^{\lambda k}\mu(k) < \infty$ for some $\lambda > 0$, which is a very particular case of $\alpha = 2$. All their results extend to our setting, let us focus on one, which is a necessary and sufficient condition for the convergence towards the Brownian snake Z^{H^α} driven by the random excursion H^α, as defined in Section 1.1.3. Recall that L is the common law of the label increments on the edges and C_{q} is the constant from (1.27).

Theorem 1.9 ([Mar20]). *Suppose that* $\mathbf{E}[L] = 0$ *and* $\sigma_L^2 := \mathbf{E}[L^2] \in (0,\infty)$*, then, jointly with* (1.27)*, the following convergence in distribution holds in the sense of finite-dimensional marginals:*

$$\left(\frac{1}{\sqrt{C_{\mathrm{q}}\sigma_L^2 n^{(\alpha-1)/\alpha}}} Z^n_{2nt}\right)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (Z^{H^\alpha}_t)_{t\in[0,1]}.$$

It holds for the uniform topology if and only if $\mathbf{P}(|L| \geq k) = o(k^{-2\alpha/(\alpha-1)})$.

The last assumption is weaker than $\mathbf{E}[L^{2\alpha/(\alpha-1)}] < \infty$ but stronger than $\mathbf{E}[L^{2\alpha/(\alpha-1)-\varepsilon}] < \infty$ for any $\varepsilon > 0$. See Figure 1.12 for one simulation falling into this framework and two outside.



(a) $L \sim \mathrm{Unif}([-1,1])$ (b) $\mathbf{P}(|L| \geq k) \sim k^{-2\alpha/(\alpha-1)}$ (c) $\mathbf{P}(|L| \geq k) \sim k^{-0.6\times\alpha/(\alpha-1)}$

Figure 1.12 – Three instances of random label processes on the same tree with $\alpha = 1{,}3$ and $n = 10\,000$ for different laws of L. Only the left one fits into the framework of Theorem 1.9 and approximates the Brownian Snake on the stable tree. The middle one converges the this snake plus vertical peaks, and the right one to only vertical peaks (and at at different scale).

As opposed to the model of the previous sections in relation with maps, for which tightness was proved in great generality and the convergence of the finite dimensional marginals was much harder to obtain, here the convergence of the marginals is a direct application of Duquesne's theorem (1.27) together with the Central Limit Theorem applied to independent L-random walks on finitely many branches and the key point here is to prove tightness. For this, we rely on Kolmogorov's criterion, which

enables one to avoid dealing with all the correlations due to the genealogy of the trees. The proof of Theorem 1.9 follows closely the paper [JM05], extending the technical estimates when needed. The key technical input is a strong control on the geometry of the trees. Precisely, although the convergence (1.27) implies that the sequence $(n^{-(\alpha-1)/\alpha} C^n_{2n\cdot})_{n\geq 1}$ is tight in the space of continuous function, we use for Theorem 1.9 the following more precise estimate on the geometry of these trees.

Lemma 1.4 ([Mar20]). *For every $\gamma \in (0, (\alpha-1)/\alpha)$, it holds that*

$$\lim_{K\to\infty} \liminf_{n\to\infty} \mathbf{P}\Big(\sup_{0\leq s\neq t\leq 1} n^{-(\alpha-1)/\alpha} \cdot \frac{|C^n_{2nt} - C^n_{2ns}|}{|t-s|^\gamma} \leq K \Big) = 1.$$

Note that the maximal exponent $(\alpha-1)/\alpha$ corresponds to the maximal exponent for which the limit process H^α is Hölder continuous [DLG02, Theorem 1.4.4]. Gittenberger [Git03] proved a similar statement in the case $\alpha = 2$, when the offspring distribution admits finite exponential moments, by means of analytic combinatoric tools, especially singularity analysis. Here we rely on the description of the contour (or rather height) process in terms of the Łukasiewicz path, namely as local times of the reversed path, and strong bounds on such walks. Both Theorem 1.9 and Lemma 1.4 have led to several applications of random walks on random trees [And18, Arc20b, DKLT22, BH22] which are of independent interest.

Janson & Marckert [JM05] also discuss the case of heavier tails for the label increments, in which case the label process converges once suitably rescaled towards a 'hairy snake' with vertical peaks. Since the limit is not a function, the convergence holds in a generalised space, namely the graph of the label process converges in the Hausdorff topology on compact sets of $[0,1]\times\mathbf{R}$. Let us refer to Figure 1.12 middle and right for simulations and to [Mar20] for precise statements.

Instead, let us only mention the following simple result, when the label increments are not centred, which extends [JM05, Theorem 8].

Theorem 1.10 ([Mar20]). *Suppose that $m := \mathbf{E}[L] \neq 0$. Then the sequence $(n^{-(\alpha-1)/\alpha} Z^n_{2n\cdot})_{n\geq 1}$ is tight in the space of continuous functions* if and only if $\mathbf{P}(|L|\geq k) = o(k^{-(\alpha-1)/\alpha})$. *In this case we have the convergence in distribution*

$$\Big(\frac{1}{C_{\mathbf{q}} n^{(\alpha-1)/\alpha}} Z^n_{2nt} \Big)_{t\in[0,1]} \xrightarrow[n\to\infty]{(d)} (m H^\alpha_t)_{t\in[0,1]},$$

jointly with (1.27).

Again, the assumption is slightly weaker than $\mathbf{E}[L^{\alpha/(\alpha-1)}] < \infty$. In the particular case when $L \geq 0$ almost surely and $m > 0$, the label increment along each edge can be interpreted as a random length so Z^n can be interpreted as the contour process of the tree T^n with such random edge-lengths. Theorem 1.10 shows that it converges towards m times the stable tree for the GHP topology, jointly with the original tree, as if the edge-lengths were fixed to m, thus answering a question of Aldous [Ald93, Section 5.3] in the case of finite-variance offspring distribution.

1.3.5 Comments and perspectives

Let us describe several directions in which we aim at continuing this line of research.

Towards Lévy maps From the point of view of maps, the biconditioned models naturally lead in some regimes to non stable Lévy excursions and motivate us to develop a theory of Lévy random maps which would appear more generally as limits of Boltzmann maps conditioned to have say n edges and sampled with a weight sequence $\mathbf{q}_n$ that may vary with n. One can construct the possible limits as in Section 1.2.4 where X is now the normalised excursion of a Lévy process with no negative jump, which can be defined under some technical assumptions [CUB11, Theorem 5] (see also the note added in proof in [UB14]). In a work in progress with Igor Kortchemski, we calculate the fractal (Hausdorff, packing, Minkowski) dimensions of the corresponding continuum looptrees and maps and we study more the (α,λ) family of maps mentioned at the end of Section 1.3.3.

Convergence of biconditioned trees Section 1.3.3 discusses the convergence of the Łukasiewicz path of random trees conditioned both by their number of vertices and leaves. In the case of the uniform distribution, Labarbe & Marckert [LM07] proved the convergence of their contour process to the Brownian excursion, and thus of the trees in the GHP topology to the Brownian CRT. For more general Bienaymé–Galton–Watson trees, controlling the Łukasiewicz path is usually the first step towards the convergence of their height process, which we aim to study more.

Multitype labelled trees Another direction would be to extend the results of Section 1.3.4 to more general label increments, which would be neither identically distributed, nor independent (for siblings, but independent from branchpoints to branchpoints). Such distributions appeared in Section 1.3.2, namely the uniform distribution on $\mathbf{B}_k^{\geq -1}$ and others appear with non bipartite maps. We believe that our method from [Mar20] allows to obtain the convergence to the Brownian snake driven by a Brownian excursion in the case $\alpha = 2$ under a '$4+\varepsilon$-moment' assumption as in [MM07, Theorem 8] who worked with finite exponential moments for the offspring distribution. We would like to push this to an optimal assumption as well as to consider the stable regimes $\alpha \in (1, 2)$, in which the details on the law of the increments would matter more.

Further, it would be nice to be able to control multitype trees in which the offspring distribution depends on the label of the vertex. In addition to applications to maps, this could be a more realistic model of branching processes, in which the environment affects the reproduction, viewing the labels as a position in space, which can be more or less fertile. Several invariance principles are known for the contour process of multitype trees [Mie08a, dR17, BO18, HS21], in which the type is viewed as the genotype, which changes from an individual to its children by possible mutation and we hope to be able to learn from these references to extend Theorem 1.9 in such a context.

Chapter 2

Local limits: A study of infinite discrete stable maps

This chapter presents the papers [BCM18, CM18a, CM20, CM21, CKM22] pertaining to the theory of local limits of random maps. It is organised as follows.

Section 2.1. We briefly describe the local topology used in this chapter and recall convergence results of stable Boltzmann maps due to Björnberg & Stefánsson [BS14] and to Stephenson [Ste18]. The aim of the next section is to study these limits. For this the key technical tool is Budd's version of the *peeling exploration* [Bud16] that we also recall. We finally present some examples of peeling procedures that we will use.

Section 2.2. We describe in this section the papers [BCM18, CM18a, CM20, CM21] in collaboration with Timothy Budd & Nicolas Curien on the infinite stable Boltzmann maps, whose large scale properties are characterised by a number $a \in (3/2, 5/2]$. We start with the two middle ones, which focus on general properties shared by any peeling process and try to answer in particular the question: how does such an exploration grow? In particular in [CM20] we argue from volume considerations that with high probability the explored region after n steps lies in an annulus between two (hulls of) balls of radius of order $n^{1/(2a-2)}$. In [BCM18] we focus on the dual map in the case $a = 2$ and prove by careful analysis of a well-chosen peeling procedure that this dual map has a so-called 'intermediate volume growth', precisely the (hull of the) ball of radius r of this dual map has volume $\exp(C\sqrt{r})$ for some model-dependent constant $C > 0$. Finally in [CM21] we study the behaviour of the simple random walk on the infinite map and on its dual. The main result shows that the walk is subdiffusive: it displaces at speed at most of order $n^{1/3}$ after n steps in all regimes $a \in (3/2, 5/2]$, which extends a previous work by Benjamini & Curien [BC13] on quadrangulations. The ideas are very different from this reference, which would only lead to the upper bound $n^{1/(2a-2)} \in [n^{1/3}, n)$. Here we flash the walk on a well-chosen subset, which is constructed by peeling, and emulate the so-called 'distances from infinity' by coming back to a finite model and looking at the distances from a large face far away.

Section 2.3. In this final section we discuss the recent work [CKM22] in collaboration with Nicolas Curien & Igor Kortchemski on a model of plane maps conditioned to have a fixed large number of vertices, edges, and faces, similar to Section 1.3.3 although we consider the uniform distribution only, on all, non necessarily bipartite, maps. We try to understand the geometry of such maps rescaled by a smaller factor than the diameter, which keeps the faces macroscopic, and see how they are put together. From a technical point of view, this work has little to do with the rest of this document, and relies on the decomposition of the map as a *kernel*, or *scheme*, in which each edge should be replaced by a tree with two marked points to recover the original map. Relying on a recent work by Budzinski [Bud21],

we ultimately prove that our random maps, once rescaled by well-chosen factor, much smaller than its diameter, converge in distribution to a semi-continuous limit, constructed by taking an infinite discrete random map, namely here the dual of the UIPT, and replacing each edge by a random continuum Brownian tree (with random volume) with two marked points.

2.1 Local limits and the peeling process

In this second chapter, we adopt a different point of view on maps as we will consider now limits of large maps without scaling distances. The limit objects are thus not continuum spaces anymore but rather infinite discrete maps embedded on the whole plane, meant to describe the microscopic behaviour of large maps. Here we shall not be interested in convergence results which already exist in the literature, instead our aim is to study the properties of the limits. Let us provide here some necessary background on this topic; as for Section 1.1, this topic was covered in a summer school in Saint-Flour, by Nicolas Curien, and we refer the reader to the associated lecture notes [Cur19].

2.1.1 Local limits of Boltzmann maps

A way to compare two finite rooted maps $\mathfrak{m}$ and $\mathfrak{m}'$ is by using the *local distance*, introduced by Benjamini & Schramm [BS01]. For every $r \geq 0$, let $\mathrm{Ball}(\mathfrak{m}, r)$ denote the ball of radius r in $\mathfrak{m}$ centred at the origin ρ of the root edge, that is the map made of all the faces of $\mathfrak{m}$ with at least one vertex at distance strictly less than r from the root vertex (when $r = 0$ we let $\mathrm{Ball}(\mathfrak{m}, r) = \{\rho\}$ be the trivial vertex map). Then the distance between two finite maps $\mathfrak{m}$ and $\mathfrak{m}'$ is given by

$$d_{\mathrm{loc}}(\mathfrak{m}, \mathfrak{m}') = \left(1 + \sup\{r \geq 0 : \ \mathrm{Ball}(\mathfrak{m}, r) = \mathrm{Ball}(\mathfrak{m}', r)\}\right)^{-1}.$$

The space of finite maps is not complete as finite maps may converge to infinite maps which can be seen as non stationary consistent sequences of finite maps $(\mathfrak{m}_n)_{n\geq 0}$ in the sense that $\mathfrak{m}_r = \mathrm{Ball}(\mathfrak{m}_n, r)$ for every $n \geq r \geq 0$. However once completed it is a Polish space, see [Cur19, Chapter 2].

Note that the complement of a ball may not be connected. We shall only deal with infinite maps which are *one-ended*, i.e. such that the connected components of the complement of any ball are all finite but one. Such maps can be embedded on the plane with no accumulation point.

Definition 2.1. If $\mathfrak{m}_\infty$ is an infinite one-ended map, we let $\mathrm{Ball}(\mathfrak{m}_\infty, r)$ denotes its ball of radius r, centred at the origin of the root edge, and $\overline{\mathrm{Ball}}(\mathfrak{m}_\infty, r)$ the *hull of this ball*, defined as the union of this ball and all the finite connected components of its complement.

As we already mentioned, Björnberg & Stefánsson [BS14] and Stephenson [Ste18] extended the local convergence of random triangulations to the UIPT of Angel & Schramm [AS03] to Boltzmann maps conditioned to be large. Recall from Section 1.3.1 the model of Boltzmann maps sampled from a weight sequence $\mathbf{q}$ and the notion of admissibility and criticality from Definition 1.2. For every $p, n \geq 1$, let $\mathbf{P}_n^{(p)}$ denote the law of such a map with perimeter $2p$ and conditioned to have n edges (we implicitly restrict ourselves to indices n for which it is well-defined).

Theorem 2.1 ([Ste18, BS14]). *If $\mathbf{q}$ is an admissible and critical weight sequence, then for every $p \geq 1$, the laws $\mathbf{P}_n^{(p)}$ converge weakly for the local topology as $n \to \infty$ to a probability distribution $\mathbf{P}_\infty^{(p)}$ on infinite one-ended maps with perimeter $2p$.*

Actually [BS14, Theorem 1.1] also treats the case of subcritical or even non admissible weight sequences, although the limits in these cases are more degenerate. On the other hand [Ste18, Theorem 6.1] deals with possibly non bipartite maps, conditioned on their number of vertices instead at this level of generality, or conditioned on their number of edges or of faces, but then assuming more regularity on the weights. In both cases, the proof use the representation of maps as labelled trees [BDFG04] as in

the first part of this document. In a nutshell, they prove the local convergence of the associated trees, then the labels are defined using local rules so it passes to the limit, and then the second key step is to prove that the construction of the maps from the labelled tree is 'local' as well, or more formally is continuous for the local topology. Let us also mention [Bud17] or [Cur19, Chapter 7] for a proof of the convergence based the peeling explorations described below.

Definition 2.2. We shall denote by $M_\infty^{(p)}$ a random infinite Boltzmann map with law $\mathbf{P}_\infty^{(p)}$, and simply $M_\infty = M_\infty^{(1)}$, which can be seen as a map without boundary by gluing together the two boundary edges.

We shall not be interested in convergence results of other models of maps, but instead study the properties of these infinite random maps M_∞. A key tool is the peeling exploration as introduced by Budd [Bud16] that we next briefly recall, following [Cur19], to which we refer for details.

2.1.2 Peeling infinite Boltzmann maps

Let $\mathfrak{m}_\infty$ be an infinite, one-ended plane map. Let us call a *submap* of $\mathfrak{m}$ a map $\overline{\mathfrak{e}}$ with a distinguished face with a simple boundary, called its *hole* and viewed as the external face, such that $\mathfrak{m}_\infty$ can be recovered by gluing a proper map with a (general) boundary inside the hole of $\overline{\mathfrak{e}}$. A *filled-in peeling process* of $\mathfrak{m}_\infty$ is then a sequence of submaps $\overline{\mathfrak{e}}_0 \subset \overline{\mathfrak{e}}_1 \subset \cdots \subset \mathfrak{m}_\infty$ constructed recursively started from $\overline{\mathfrak{e}}_0$ being simply the 2-gon containing the root edge in the following way. At each step $n \geq 0$, we select an edge $\mathscr{A}(\overline{\mathfrak{e}}_n)$ (the *peel edge*) on the boundary of the hole of $\overline{\mathfrak{e}}_n$ and reveal the face lying next to it in the 'unexplored region'. Two cases may appear and are illustrated in Figure 2.1.

- Either the peel edge is incident to a new face in $\mathfrak{m}_\infty$ of degree $2k \geq 2$, then $\overline{\mathfrak{e}}_{n+1}$ is obtained from $\overline{\mathfrak{e}}_n$ by gluing this face on the peel edge without performing any other identification. This event is called event of type C_k.
- Or the peel edge is incident to another face of $\overline{\mathfrak{e}}_n$ in the map $\mathfrak{m}_\infty$, in which case we perform the identification of the two boundary edges of $\overline{\mathfrak{e}}_n$. When doing so, the hole of $\overline{\mathfrak{e}}_n$ of perimeter say 2ℓ is split into two holes of perimeter $2\ell_1$ and $2\ell_2$ with $\ell_1 + \ell_2 = \ell - 2$. Since $\mathfrak{m}_\infty$ is one-ended, then only one of these holes contains an infinite region in $\mathfrak{m}_\infty$. We then fill-in the finite hole with the corresponding map inside $\mathfrak{m}_\infty$ to obtain $\overline{\mathfrak{e}}_{n+1}$. We speak of event of type G_{*,ℓ_1} or $\mathsf{G}_{\ell_2,*}$ depending whether the finite hole is on the left or on the right of the peel edge.

Let us stress that the choice of the peel edge at each step is given by a peeling algorithm $\mathscr{A}$ which may be deterministic or may depend on another source of randomness as long as it is independent of the unrevealed part, see Section 2.1.4 for examples. We sometimes speak of 'Markovian algorithm', this denomination comes from the following result due to Budd [Bud16], see also [Cur19, Chapter 7].

Theorem 2.2 ([Bud16]). *Sample $M_\infty^{(p)}$ from $\mathbf{P}_\infty^{(p)}$ using an admissible and critical weight sequence and let $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ be any (Markovian) peeling exploration.*

(i) Then $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ is a Markov chain whose transition probabilities are universal and given as follows: if the perimeter of $\overline{\mathfrak{e}}_n$ equals $2\ell \geq 2$, then the events C_k, $k \geq 1$ and $\mathsf{G}_{*,k}$, $\mathsf{G}_{k,*}$, $0 \leq k \leq \ell - 2$ occur with conditional probability:

$$\mathbf{P}(\mathsf{C}_k) = \frac{h^\uparrow(\ell + k - 1)}{h^\uparrow(\ell)} \nu_{\mathbf{q}}(k-1) \qquad \text{and} \qquad \mathbf{P}(\mathsf{G}_{*,k}) = \mathbf{P}(\mathsf{G}_{k,*}) = \frac{1}{2}\frac{h^\uparrow(\ell - k - 1)}{h^\uparrow(\ell)} \nu_{\mathbf{q}}(-k-1),$$

where $\nu_{\mathbf{q}}$ is a probability distribution on $\mathbf{Z}$ and $h^\uparrow$ is given for every $k \geq 0$ by:

$$h^\uparrow(k) = 2k \cdot 2^{-2k} \binom{2k}{k} \mathbf{1}_{\{k\geq 1\}}. \tag{2.1}$$

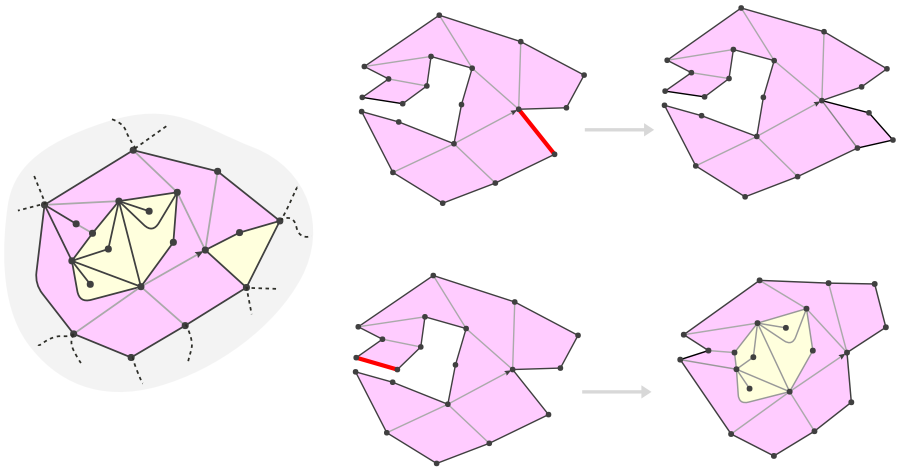


Figure 2.1 – Illustration of the filled-in peeling process. On the left we have explored a certain region $\overline{\mathfrak{e}}_n \subset \mathfrak{m}_\infty$ corresponding to the faces in pink glued by the edges in gray. Depending on the edge to peel at the next step we may end-up either with an event of type C_2 (top figures), or an event of type $\mathsf{G}_{3,*}$ (bottom figures).

(ii) Moreover, in the case of the events $\mathsf{G}_{*,k}$ and $\mathsf{G}_{k,*}$, the map that fills-in the hole which is created is an independent finite Boltzmann map with boundary length $2k$ and free volume, i.e. with law $\mathbf{P}^{(k)}$.

(iii) Finally, the so-called spatial Markov property *holds: conditionally given the explored region, say $\overline{\mathfrak{e}}_n$, with perimeter 2ℓ, the unexplored region is independent and has the law $\mathbf{P}_\infty^{(\ell)}$.*

Let us refer to [Cur19, Chapter 5] for a definition of $\nu_{\mathbf{q}}$, where it is also proved (originally by Budd [Bud16]) that $\mathbf{q}$ is critical if and only if the $\nu_{\mathbf{q}}$-random walk oscillates, if and only if $h^{\uparrow}$ is harmonic for the $\nu_{\mathbf{q}}$-random walk killed upon entering $\mathbf{Z}_{\leq 0}$. As a direct corollary, denote by $P_n = \frac{1}{2}|\partial\overline{\mathfrak{e}}_n|$ the half-perimeter of $\overline{\mathfrak{e}}_n$ and $V_n = |\overline{\mathfrak{e}}_n|$ its volume, say as the number of inner vertices. Then the pair $(P_n, V_n)_{n\geq 0}$ is a Markov chain whose law does not depend on the peeling algorithm, and more precisely:

- The sequence $(P_n)_{n\geq 0}$ has the same law as the Doob h-transform of the $\nu_{\mathbf{q}}$-random walk with the harmonic function $h^{\uparrow}$ from (2.1), which is a way to make sense of the law of such a walk conditioned to stay positive forever, starting from $P_0 = p$ under $\mathbf{P}_\infty^{(p)}$.
- Conditional on $(P_n)_{n\geq 0}$, the random variables $(V_{i+1} - V_i)_{i\geq 0}$ are independent, each $V_{i+1} - V_i$ is null if $P_i - P_{i+1} - 1 \leq 0$, otherwise it is distributed as the volume of a map sampled from $\mathbf{P}^{(k)}$ with $k = P_i - P_{i+1} - 1$.

2.1.3 Stable-type weight sequences

The perimeter and volume processes are key tools in the study of M_∞. Let us next recall an invariance principle for these processes for so-called discrete stable maps. Recall the law $\nu_{\mathbf{q}}$ from in Theorem 2.2.

Definition 2.3. Fix $a \in [3/2, 5/2]$; a sequence $\mathbf{q}$ is said to be *of type a* if it is admissible and there exists a constant $\mathsf{p}_{\mathbf{q}} > 0$ such that

$$\nu_{\mathbf{q}}(-k) \underset{k\to\infty}{\sim} \mathsf{p}_{\mathbf{q}} \cdot k^{-a} \qquad \text{and} \qquad \nu_{\mathbf{q}}([k,\infty)) \underset{k\to\infty}{\sim} \mathsf{p}_{\mathbf{q}} \frac{\cos(a\pi)}{a-1} \cdot k^{-(a-1)}, \tag{2.2}$$

where the second equivalent should be understood as $\nu_{\mathbf{q}}([k,\infty)) \ll k^{-(a-1)}$ when $a = 3/2$ or $a = 5/2$.

Remark 2.1. Let us link this definition with that of Section 1.3:

- $\mathbf{q}$ is of type $a = 3/2$ if and only if it is subcritical in the sense of Definition 1.2,
- $\mathbf{q}$ is of type $a \in (3/2, 5/2]$ if and only if it is critical and discrete α-stable in the sense of Definition 1.3 with $\alpha = a - 1/2$.

See [Cur19, Chapter 5] for the proof and more equivalent conditions. One could work out the general case of stable domain of attractions, this would only replace constants by slowly varying functions.

When $\mathbf{q}$ is of type $a \in [3/2, 5/2]$, the law $\nu_{\mathbf{q}}$ belongs to the domain of attraction of an $(a-1)$-stable law, and is centred when $a > 2$, so once normalised by a factor of order $n^{1/(a-1)}$, it converges in distribution to the $(a-1)$-stable Lévy process $\mathrm{Y}_a = (\mathrm{Y}_a(t))_{t\geq 0}$ with Lévy measure

$$\Pi(\mathrm{d}x) = \cos(a\pi)\frac{\mathrm{d}x}{x^a}\mathbf{1}_{\{x>0\}} + \frac{\mathrm{d}x}{|x|^a}\mathbf{1}_{\{x<0\}}.$$

Note the three particular cases: when $a = 5/2$ this is a centred 3/2-stable process with no positive jump, when $a = 2$ this is the symmetric Cauchy process, and finally when $a = 3/2$ this is the negative of a 1/2-stable subordinator. Actually in the case $a = 2$, often left aside in the literature, thanks to the tail behaviour (2.2) there exists a sequence $(b_n)_{n\geq 1}$ such that the walk rescaled by n and shifted by $b_n t$ converges to Y_2 but one needs an extra argument to prove that $b_n = 0$, which is provided in [BCM18].

From now on we exclude the subcritical case $a = 3/2$. Recall that the half-perimeter of a peeling exploration has the law of a $\nu_{\mathbf{q}}$-random walk conditioned to stay positive. Then by the results of Caravenna & Chaumont [CC08], the latter converges in distribution to $\mathrm{Y}_a^{\uparrow}$, the version of Y_a conditioned to stay positive, which can be similarly defined via the Doob h-transform, with the harmonic function $u \mapsto \sqrt{u}$. On the other hand, recall that the volume process can be written as the cumulative sum of independent random variables whose law depends on the values of the negative increment of the perimeter, then one can define similarly a 'continuum volume process' $\mathcal{V}(\mathrm{Y}_a^{\uparrow})$, see [BC17] or [Cur19, Chapter 10], and prove the following invariance principle.

Theorem 2.3 ([BC17, BCM18]). *Suppose that* $\mathbf{q}$ *is of type* $a \in (3/2, 5/2]$. *Then there exists two constants* $\mathsf{p}_{\mathbf{q}}, \mathsf{b}_{\mathbf{q}} > 0$ *such that the convergence in distribution*

$$\left(n^{-\frac{1}{a-1}} P_{[nt]}, n^{-\frac{a-1/2}{a-1}} V_{[nt]}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\mathrm{Y}_a^{\uparrow}(\mathsf{p}_{\mathbf{q}} t), \mathsf{b}_{\mathbf{q}} \mathcal{V}(\mathrm{Y}_a^{\uparrow})(\mathsf{p}_{\mathbf{q}} t)\right)_{t\geq 0},$$

holds for the Skorokhod topology.

The argument was initiated in [CLG17] for triangulations and quadrangulations and generalised in [BC17, Theorem 3.6] in the case $a \in (3/2, 2) \cup (2, 5/2)$, but the argument extends to $a = 5/2$; the case $a = 2$ is treated in [BCM18] as we mentioned above.

2.1.4 Examples of peeling algorithms

The strength of peeling explorations is that all peeling algorithms have the same law so universal estimates can be used as we just saw, and on the other hand many algorithms can be used to study different properties of the maps. Let us describe very briefly a few of them and refer to [Cur19] for more details and more peeling algorithms. We shall use these algorithms in the next section.

Peeling by layers on the dual The algorithm $\mathcal{A}_{\mathrm{dual}}$, illustrated in Figure 2.2, is designed to reveal the dual balls centred at the root face one after the other. This algorithm starts as always from the 2-gon and then 'turns around' the boundary and peel successively the edges adjacent to a face whose dual graph distance to the root face is minimal. At the first time at which no edge on the boundary is adjacent to a face at dual distance R from the root face, the piece revealed is then equal to the hull of the ball of radius R in the dual map $M_\infty^\dagger$.

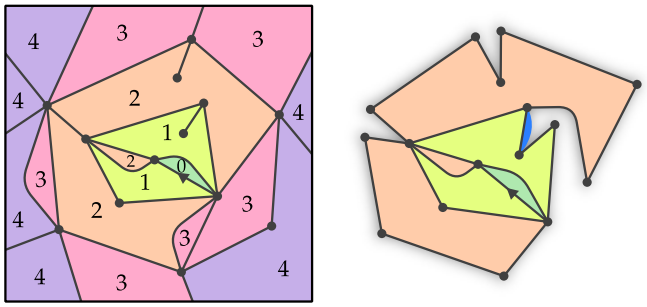


Figure 2.2 – Left: A portion of an infinite planar map with faces labelled according to dual graph distance to the root face. Right: A possible state of the peeling with algorithm $\mathcal{A}_{\text{dual}}$; the next edge to peel is indicated in blue.

Peeling by layers on the primal One can adapt the previous algorithm to define the algorithm $\mathcal{A}_{\text{metric}}$, illustrated in Figure 2.3, which discovers one after the other the hull of the balls of the original map centred at the origin ρ of the root edge. This is done by now turning around the origin to reveal all its neighbours, then continue with each of them and then their neighbours etc. More formally, we always peel an edge $\mathcal{A}_{\text{metric}}(\bar{\mathfrak{e}}_n)$ with an end point at minimal distance to ρ. Note that as opposed to $\mathcal{A}_{\text{dual}}$, the distances of the vertices along the boundary of $\bar{\mathfrak{e}}_n$ to the origin may here differ in $\bar{\mathfrak{e}}_n$ and in the true map since a vertex may seem far in $\bar{\mathfrak{e}}_n$ but may come close after a gluing event. However, it is easy to check that they agree for the vertices at minimal graph distance from the origin.

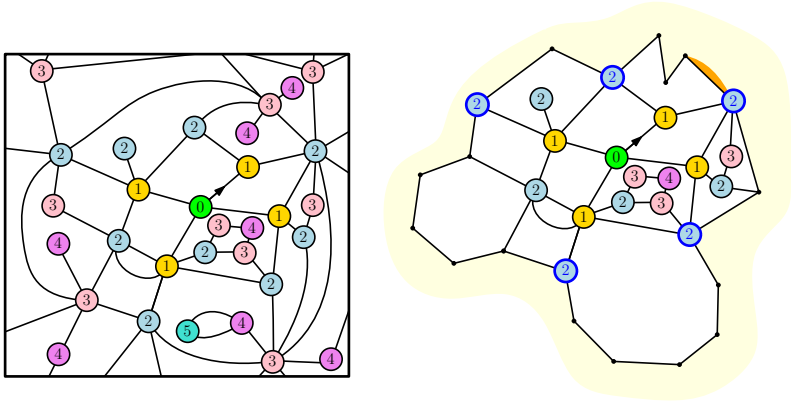


Figure 2.3 – Illustration of the algorithm $\mathcal{A}_{\text{metric}}$. Vertices are labelled by their distance to the origin. On the right a current state $\bar{\mathfrak{e}}_n$ of the exploration, the vertices on the boundary with minimal distance to the origin have the same labels in $\bar{\mathfrak{e}}_n$ and in the underlying map.

Uniform peeling An obvious random algorithm $\mathcal{A}_{\text{unif}}$ consists in simply choosing at each step the peel edge uniformly at random on the boundary of the explored region. This is actually related to an interesting model which is sometimes called Eden model, which corresponds to first passage percolation on the dual map. Indeed, equip each edge of M_∞ with i.i.d. exponential clocks with rate 1, independent of M_∞ and at each step of the exploration, peel the first edge on the boundary whose clock rings, then this provides a uniform random choice by standard properties of the exponential random variables.

Peeling along simple random walks Another interesting random algorithm $\mathcal{A}_{\text{SRW}^\dagger}$ consists in letting a simple random walk evolve on the dual map. The latter starts from the root face and first wants to cross some edge, then we peel this edge. More generally, we peel an edge whenever the walk wants to go outside the hull of its previous trajectory, as in Figure 2.4.

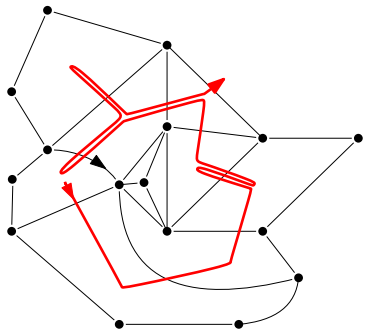


Figure 2.4 – Illustration of the algorithm $\mathcal{A}_{\mathrm{SRW}^\dagger}$: the walk on the dual map is about to leave the hull of its past trajectory, calling for a new peeling step.

One can adapt it to the algorithm $\mathcal{A}_{\mathrm{SRW}}$ which follows a walk on the original map, starting from the origin of the root edge. Here whenever the walk sits on the boundary of the explored region, we reveal all its neighbours as in the algorithm $\mathcal{A}_{\mathrm{metric}}$ before moving to the next position of the walk.

2.2 Properties of the infinite stable maps

We describe in this section the papers [BCM18, CM18a, CM20, CM21] in collaboration with Timothy Budd and Nicolas Curien on the infinite Boltzmann maps M_∞ with type $a \in (3/2, 5/2]$. It strongly relies on the previous section.

2.2.1 Generalities on the peeling of stable maps

In the papers [CM18a, CM20] we study general properties of the exploration processes of M_∞, with in mind the application to the behaviour of the simple random walk on these graphs which is further detailed in the Section 2.2.3.

How long does the root edge remain exposed?

In the short paper [CM18a] we asked the following question: if $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ is any (filled-in Markovian) peeling process of M_∞, then how long does it take for the root edge $\vec{e}$ to be 'swallowed' by $(\overline{\mathfrak{e}}_n)_{n\geq 0}$, i.e. not to lie on the boundary of the explored region anymore? We answered in the case of bounded face degrees (so $a = 5/2$) for simplicity. Let us mention that [CM18a] actually considers general, not necessarily bipartite, maps.

Theorem 2.4 ([CM18a]). *Let M_∞ be an infinite critical Boltzmann planar map with bounded face degrees and let $(\overline{\mathfrak{e}}_n)_{n\geq 0}$ be any peeling process of M_∞. We have as $n \longrightarrow \infty$:*

$$\mathbf{P}(\vec{e} \in \partial\overline{\mathfrak{e}}_n) \leq n^{-2c/3+o(1)}, \tag{2.3}$$

where c is the positive solution to

$$\frac{4\pi}{3} = \int_0^1 x^{c-1}(1-x)^{1/2}\,\mathrm{d}x \cdot \int_0^{1/2} x^{c+1/2}(1-x)^{-5/2}\,\mathrm{d}x.$$

Remark 2.2. The constant c is approximately

$$c \approx 0{,}128\,312\,351\,417\,832.$$

We provided in [CM18a] more digits, simply obtained by brute calculation, but Cyril Banderier wrote us in November 2021 to correct this longer approximation using Beta integral and hypergeometric functions. None of us found any closed formula for c.

To prove (2.3), simply observe that the root edge lies on the boundary of $\overline{\mathfrak{e}}_n$ if and only if it has not been peeled and for each exploration step before time n which led to the gluing of two edges on the boundary, the part of the boundary swallowed during this operation did not contained the root edge. By taking into account only the steps that swallowed at least half of the boundary, and decomposing into scales according to the value of the perimeter, we obtain (2.3) where $c > 0$ is defined implicitly by

$$\exp(-c\log 2) = \mathbf{E}_1\left[\exp\left(-\#\left\{0 \le t \le \theta(2) :\ \Delta \mathrm{Y}^{\uparrow}_{5/2}(t) < -\frac{1}{2}\mathrm{Y}^{\uparrow}_{5/2}(t-)\right\}\log 2\right)\right]$$

where $\theta(x) = \inf\{t \ge 0 :\ \mathrm{Y}^{\uparrow}_{5/2}(t) \ge x\}$ and we recall that $\mathrm{Y}^{\uparrow}_{5/2}$ is the limit of the perimeter process from Theorem 2.3, which in this case is a 3/2-stable Lévy process with no positive jumps conditioned to stay positive. The characterisation of c as in the theorem then follows from the Lamperti transform and a kind of compensation formula.

It can be observed that the exponent $2c/3$ is intimately related to the $n^{2/3}$ growth of the perimeter process as in Theorem 2.3 and it is optimal in the sense that no smaller exponent can hold for all peeling explorations. Of course, some peeling explorations swallow the root edge much more rapidly (up to one step for a peeling that starts by exploring the root edge, such as the one exploring the metric balls). In another direction, we believe that Theorem 2.4 holds for any Boltzmann map with type $a = 5/2$, although one must be more careful if the face degrees (and thus the positive increments of the perimeter process) are not bounded. For maps with type $a \in (3/2, 5/2)$, in addition to this difficulty, the limit of the perimeter process has both positive and negative jumps, so we also loose the Lamperti transformation and the link with positive self-similar Markov processes which allowed us to calculate c.

Explorations are roundish

From a more general perspective, in [CM20] we asked about the behaviour of the entire explored region in a peeling process, in the general framework of stable weights of type $a \in (3/2, 5/2]$ in the sense of Definition 2.3. In a sense, any peeling exploration of M_∞ grows like the hull of a metric ball.

Theorem 2.5 ([CM20]). *Fix a critical weight sequence* $\mathbf{q}$ *of type* $a \in (\frac{3}{2}, \frac{5}{2}]$. *For any* $\varepsilon > 0$, *there exist* $0 < c_\varepsilon < C_\varepsilon < \infty$ *such that for any peeling exploration* $(\overline{\mathfrak{e}}_n)_{n\ge 0}$, *we have for every* n *large enough:*

$$\mathbf{P}\left(\overline{\mathrm{Ball}}\left(M_\infty, c_\varepsilon n^{\frac{1}{2(a-1)}}\right) \subset \overline{\mathfrak{e}}_n \subset \overline{\mathrm{Ball}}\left(M_\infty, C_\varepsilon n^{\frac{1}{2(a-1)}}\right)\right) \ge 1-\varepsilon.$$

This shows that any peeling exploration eventually discovers the full map and even provides the speed at which it reveals the map.

Remark 2.3. We also prove in [CM20, Proposition 2] that hulls of balls are not deep in the sense that for any $\varepsilon > 0$, there exists $K_\varepsilon \in (0, \infty)$ such that for every r large enough,

$$\mathbf{P}\left(\max\{d_{\mathrm{gr}}(\rho, u);\ u \in \overline{\mathrm{Ball}}(M_\infty, r)\} \le K_\varepsilon r\right) \ge 1-\varepsilon.$$

In the upper bound in Theorem 2.5, one can thus replace the hull of the ball by the true ball, with a larger constant. This shows that the radius of the explored region grows exactly like order $n^{\frac{1}{2(a-1)}}$.

In order to prove Theorem 2.5, denote by D_n^- and D_n^+ the smallest and the largest distance *in the whole map* to the origin vertex ρ of a vertex on the boundary $\partial\overline{\mathfrak{e}}_n$. With this definition, we have

$$\overline{\mathrm{Ball}}(M_\infty, D_n^- - 1) \subset \overline{\mathfrak{e}}_n \subset \overline{\mathrm{Ball}}(M_\infty, D_n^+ + 1),$$

so Theorem 2.5 consists in showing that with probability $1-\varepsilon$, we have both

$$D_n^- \ge c_\varepsilon n^{\frac{1}{2(a-1)}} \qquad \text{and} \qquad D_n^+ \le C_\varepsilon n^{\frac{1}{2(a-1)}}.$$

The converse bounds hold by simply comparing the volume of $\overline{\mathfrak{e}}_n$, which is of order $n^{\frac{a-1/2}{a-1}}$ by Theorem 2.3, and that of the hull of the ball of radius r, which is of order r^{2a-1} [CM20, Proposition 2].

The upper bound on D_n^+ is basically due to Benjamini & Curien [BC13] who considered quadrangulations, but their arguments apply to the general setting after extending some technical estimates. Using the aforementioned upper bound on D_n^-, it suffices to upper bound $D_n^+ - D_n^-$. Now this is smaller than the maximal distance between two points on the boundary of $\overline{\mathfrak{e}}_n$ *in the complement map* $M_\infty \setminus \overline{\mathfrak{e}}_n$ that remains to be explored. Recall the spatial Markov property from Theorem 2.2: conditionally on the perimeter $p = |\partial\overline{\mathfrak{e}}_n|/2$, this unexplored region is independent of $\overline{\mathfrak{e}}_n$ and has precisely the law of an infinite Boltzmann map with a boundary $M_\infty^{(p)}$. The greatest distance between two points on the boundary of such a map grows like $\sqrt{p}$ by [CM20, Proposition 4], and since we know the behaviour of this perimeter by Theorem 2.3, then this shows that $D_n^+ - D_n^-$ grows at most like $(n^{\frac{1}{a-1}})^{\frac{1}{2}} = n^{\frac{1}{2(a-1)}}$.

For the lower bound on D_n^-, we argue from volume considerations. The main idea being that if D_n^- is small, then a lot of gluing operations during the peeling exploration will accumulate too much volume near the origin of M_∞. Indeed, each time such a gluing swallows a point on the boundary much closer than $n^{\frac{1}{2(a-1)}}$ from the origin, a portion of the finite map that fills the hole is added to the hull of the ball of radius $n^{\frac{1}{2(a-1)}}$. See Figure 2.5 for an illustration. We provide in [CM20, Lemma 1] a lower bound on the volume of this portion which shows that if this occurs too many times, then the volume of the hull of the ball of radius $n^{\frac{1}{2(a-1)}}$ exceeds a large constant times $n^{\frac{2a-1}{2(a-1)}}$ which has a small probability to occur [CM20, Proposition 2]. On the other hand, as long as there exists a point much closer than $n^{\frac{1}{2(a-1)}}$ from the origin, it always has a positive probability (bounded from below independently of n) to be swallowed in the next peeling step. This argument leads to the conclusion that no such point can remain for n large as claimed in Theorem 2.5.

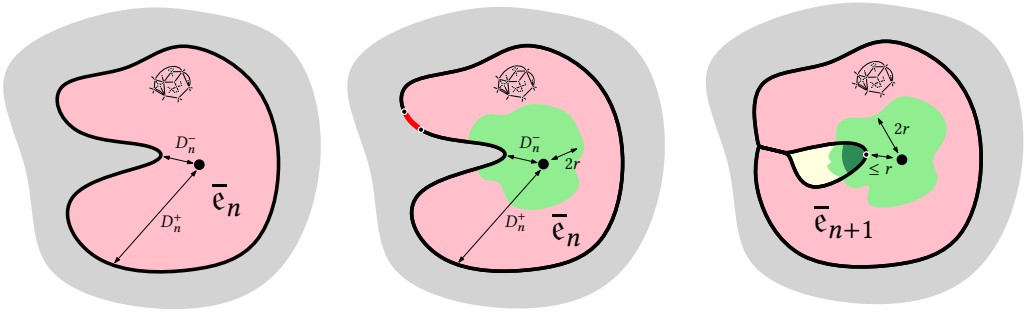


Figure 2.5 – Illustration of the proof of the lower bound on D_n^-: on the middle figure, the red edge is the one to peel and the green region is $\overline{\mathrm{Ball}}(M_\infty, 2r) \cap \overline{\mathfrak{e}}_n$. If we glue two edges and swallow a part of the boundary containing a point at minimal distance from the origin $D_n^- < r$ then we add at least to the previous ball that of radius r in the map filling in the hole centred at this point (in dark green on the rightmost figure).

We should point out that all the technical estimates on the maps in [CM20] are obtained using the correspondence with labelled trees (that from [BDFG04]) discussed in the first part of this document. For example the $\sqrt{p}$ growth of the distances on the boundary corresponds to the diffusive scaling of the geometric random bridge in Lemma 1.2 that encodes the label variations on the loop of length p coding the boundary in the looptree; also the r^{2a-1} volume growth for the (hulls of) balls is consistent with the scaling factor $n^{1/(2\alpha)} = n^{1/(2a-1)}$ in Section 1.3.2.

2.2.2 The dual stable maps and the case $a = 2$

Instead of considering the stable Boltzmann maps M_∞, Budd & Curien [BC17] studied their dual $M_\infty^\dagger$, which are maps with a heavy tailed *vertex degree* distribution. They proved that the so-called *dense* regime $a \in (3/2, 2)$ and *dilute* regime $a \in (2, 5/2)$ are very different and left open the middle case $a = 2$

which we studied together in [BCM18], showing an intermediate behaviour, in between the two others. Let us present the results of these two papers for the dual graph metric first, and then the (actually simpler to study) first passage percolation (FPP) on the dual graph. In both cases, the study strongly relies on peeling explorations.

The dual graph distances: peeling by layer

Let $\overline{\mathrm{Ball}}(M_\infty^\dagger, r)$ denote the ball of radius r in the dual map $M_\infty^\dagger$, to which we add as previously the finite connected components of its complement. We let $|\overline{\mathrm{Ball}}(M_\infty^\dagger, r)|$ denote its volume, defined as its number of inner vertices, and $|\partial\overline{\mathrm{Ball}}(M_\infty^\dagger, r)|$ its perimeter. Budd & Curien [BC17] proved a phase transition, namely that for any $a > 2$, these quantities grow polynomially, and precisely the pair of rescaled processes

$$\left(n^{-\frac{1}{a-2}}|\partial\overline{\mathrm{Ball}}(M_\infty^\dagger, \lfloor nt\rfloor)|, n^{-\frac{a-1/2}{a-2}}|\overline{\mathrm{Ball}}(M_\infty^\dagger, \lfloor nt\rfloor)|\right)_{t\geq 0}$$

converges in distribution in the Skorokhod topology, to a limit pair simply obtained by a random time change of that in Theorem 2.3, which is reminiscent of the Lamperti transform (although here the Lévy process may have positive jumps). On the other hand, for any $a < 2$, the perimeter and volume grow exponentially fast, namely there exists a constant $c_a \in (0, \infty)$ such that

$$\frac{1}{r}\log|\partial\overline{\mathrm{Ball}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} c_a \qquad \text{and} \qquad \frac{1}{r}\log|\overline{\mathrm{Ball}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} (a-1/2)\,c_a. \tag{2.4}$$

It is worth noting that the constant only depends on a and not precisely on the weight sequence $\mathbf{q}$; it is not explicit, see [BC17, Lemma 5.6].

When $a = 2$, we established in [BCM18] an *intermediate* volume growth, exponential in the square-root of the radius, and now the constants are explicit.

Theorem 2.6 ([BCM18]). *If* $\mathbf{q}$ *is critical of type* $a = 2$*, then*

$$\frac{1}{\sqrt{r}}\log|\partial\overline{\mathrm{Ball}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} \pi\sqrt{2} \qquad \textit{and} \qquad \frac{1}{\sqrt{r}}\log|\overline{\mathrm{Ball}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} \frac{3\pi}{\sqrt{2}}.$$

The idea is to apply the general invariance principle from Theorem 2.3 to the peeling algorithm $\mathcal{A}_{\mathrm{dual}}$ described in Section 2.1.4, which discovers the hull of the dual balls layer by layer. For every $n \geq 0$, let H_n denote the smallest graph distance in the dual map between the root face and a face in $\overline{\mathfrak{e}}_n$ adjacent to an edge on its boundary. Note that by construction, all such 'boundary faces' lie at dual graph distance either H_n or $H_n + 1$ from the root face. The key point is then to control the growth of H_n and Theorem 2.6 follows from Theorem 2.3 and the convergence in probability:

$$\frac{H_n}{(\log n)^2} \xrightarrow[n\to\infty]{\mathbf{P}} \frac{1}{2\pi^2}, \tag{2.5}$$

which is proved in [BCM18, Proposition 4]. Indeed, this implies that the time needed to reach dual distance r in this process, i.e. to discover $\overline{\mathrm{Ball}}(M_\infty^\dagger, r)$, is approximately $\exp(\sqrt{2\pi^2 r})$ and according to Theorem 2.3, at this time, the perimeter and volume of the exploration are approximately $\exp(\sqrt{2\pi^2 r})\mathrm{Y}_2^\uparrow(\mathsf{p}_\mathbf{q})$ and $\exp(\sqrt{2\pi^2 r})^{3/2}\mathsf{b}_\mathbf{q}\mathcal{V}(\mathrm{Y}_2^\uparrow)(\mathsf{p}_\mathbf{q})$ respectively.

For a heuristic argument towards (2.5), recall that the peeling algorithm $\mathcal{A}_{\mathrm{dual}}$ 'turns around' the boundary in clockwise order to discover, layer after layer the faces of M_∞ by increasing dual graph distance and we aim at determining at which speed. Call the *height* of a boundary edge of $\overline{\mathfrak{e}}_n$ the dual distance to the root face of the edge incident to it in $\overline{\mathfrak{e}}_n$. Then in the algorithm $\mathcal{A}_{\mathrm{dual}}$, at every step n, the picture is the following: either all the edges have the same height H_n and we just discovered $\overline{\mathrm{Ball}}(M_\infty^\dagger, H_n)$, or there is a segment of consecutive edges on the boundary at height H_n and another one at height $H_n + 1$ and the next edge to peel is the one at height H_n immediately to the right of an edge at height $H_n + 1$. Let R_n denote the number of remaining edges at height H_n.

Now let us peel this edge: if we discover a new face or get glued to a face on the left at height $H_n + 1$, then R_n decreases by 1, but if we get glued to a face on the right at height H_n, then it decreases by the size of the hole we swallow (plus two). One should not forget the case where we swallow all the edges at height H_n, but let us neglect this. Using the explicit transition probabilities from Theorem 2.2, we can determine the law of R_n and see that, in our Cauchy-type tails regime, when starting from p, it decreases at speed $p \log p$. Since P_n is of order n, then this suggests that the time needed to complete a layer around $\overline{\mathfrak{e}}_n$ is of order $n/\log n$. Hence, the average growth of H_n at each step is order $\log n/n$, so finally H_n is of order $\log^2 n$. Turning this sketch into a rigorous proof is however technical and [BCM18, Section 4] is entirely devoted to this.

Comparison with the original graph distance. Let us view the dual ball $\mathrm{Ball}(M_\infty^\dagger, r)$ as the subset of vertices of M_∞ which are incident to a face at dual graph distance from the root face less than r, and as always add all the finite regions of the complement to get $\overline{\mathrm{Ball}}(M_\infty^\dagger, r)$. By applying the general result of Theorem 2.5 to the peeling by layer and combining with the volume growth of the dual balls presented above, we deduce the following comparison between the dual and original balls.

Corollary 2.1 ([CM20]). *Fix a critical weight sequence* $\mathbf{q}$ *of type* $a \in (\frac{3}{2}, \frac{5}{2}]$*; there exists* $c_a > 0$ *(from* (2.4)*) such that the following holds: For every* $\varepsilon > 0$*, there exist* $0 < c_\varepsilon < C_\varepsilon < \infty$ *such that for every* r *large enough, we have*

$$\begin{aligned}
\overline{\mathrm{Ball}}(M_\infty, c_\varepsilon r^{\frac{1}{2a-4}}) &\subset \overline{\mathrm{Ball}}(M_\infty^\dagger, r) \subset \overline{\mathrm{Ball}}(M_\infty, C_\varepsilon r^{\frac{1}{2a-4}}) && \text{when} \quad a \in (2; 5/2],\\
\overline{\mathrm{Ball}}(M_\infty, \mathrm{e}^{\pi\sqrt{r/2}(1-\varepsilon)}) &\subset \overline{\mathrm{Ball}}(M_\infty^\dagger, r) \subset \overline{\mathrm{Ball}}(M_\infty, \mathrm{e}^{\pi\sqrt{r/2}(1+\varepsilon)}) && \text{when} \quad a = 2,\\
\overline{\mathrm{Ball}}(M_\infty, \mathrm{e}^{c_a(1-\varepsilon)r}) &\subset \overline{\mathrm{Ball}}(M_\infty^\dagger, r) \subset \overline{\mathrm{Ball}}(M_\infty, \mathrm{e}^{c_a(1+\varepsilon)r}) && \text{when} \quad a \in (3/2; 2),
\end{aligned}$$

with probability at least $1 - \varepsilon$.

Observe that $2a - 4 = 1$ when $a = 5/2$; in the case of triangulations, it is known more precisely that the distances on the primal and dual are in fact asymptotically proportional [CLG19].

The first-passage percolation: uniform peeling

Budd & Curien [BC17] also studied the first passage percolation (FPP) on the dual map $M_\infty^\dagger$ by attributing to each edge of this dual map a random length which are i.i.d. exponential random variables with mean 1. We shall denote by $\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, r)$ the corresponding hull of ball of radius $r \in \mathbf{R}_+$. They exhibited an ever more drastic phase transition at $a = 2$ for their perimeter and volume. For any $a > 2$, these two quantities again grow polynomially, with the same exponent, and precisely the pair of rescaled processes

$$\left(n^{-\frac{1}{a-2}} |\partial\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, \lfloor nt \rfloor)|, n^{-\frac{a-1/2}{a-2}} |\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, \lfloor nt \rfloor)|\right)_{t \geq 0}$$

converges in distribution in the Skorokhod topology, to the same limit in law as for the dual balls, up to a deterministic linear time change. On the other hand, for any $a < 2$, the balls grow at infinite speed in the sense that the infimum of the FPP-length over all infinite paths in $M_\infty^\dagger$ has finite expectation! Again the middle case $a = 2$ lies strictly in between and now the growth is exponential in the radius. Note that as opposed to Theorem 2.6, the constant is not universal anymore but depends on $\mathsf{p}_\mathbf{q}$ from (2.2).

Theorem 2.7 ([BCM18]). *If* $\mathbf{q}$ *is critical of type* $a = 2$ *then*

$$\frac{1}{r} \log |\partial\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} \pi^2 \mathsf{p}_\mathbf{q} \qquad \text{and} \qquad \frac{1}{r} \log |\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, r)| \xrightarrow[r\to\infty]{\mathbf{P}} \frac{3}{2}\pi^2 \mathsf{p}_\mathbf{q}.$$

As in the case of dual distances, the idea is to apply Theorem 2.3 to a well-chosen peeling exploration and to control its growth. In this case, the exploration is given by the FPP itself via the algorithm $\mathcal{A}_{\mathrm{unif}}$: imagine the dual edges as pipes and water flowing from the root face at unit speed, then the first edge to peel is the edge on the root face whose dual edge is the shortest, so its other extremity is the first dual

vertex after the origin to be wet. Then at any time $n \geq 1$, conditionally given $\bar{\mathfrak{e}}_n$, the next edge to peel is the one for which the remaining dry part is the shortest, which are again i.i.d. exponentials by lack of memory. A bit more formally, the process $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, t))_{t\geq 0}$ admits jump times $0 = T_0 < T_1 < \cdots$ and the peeling exploration $(\bar{\mathfrak{e}}_n)_{n\geq 0}$ is the sequence $(\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, T_n))_{n\geq 0}$. Then Theorem 2.7 follows exactly as above from Theorem 2.3 and the approximation given in [BCM18, Proposition 3]:

$$\mathbf{E}\left[\left|\frac{T_n}{\log n} - \frac{1}{\pi^2 \mathsf{p_q}}\right|\right] \underset{n\to\infty}{\longrightarrow} 0, \tag{2.6}$$

which shows that for large r, it takes about $\exp(\pi^2 \mathsf{p_q} r)$ steps to discover $\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, r)$.

By standard properties of the exponential distribution, conditionally on the exploration, the random variables $T_{n+1} - T_n$ are independent and distributed respectively according to the exponential law with parameter $|\partial\overline{\mathrm{Ball}}^{\mathrm{fpp}}(M_\infty^\dagger, T_n)| = 2P_n$. Alternatively, we may write these times in the form $T_n = \sum_{i=0}^{n-1} \frac{\gamma_i}{2P_i}$, where $(\gamma_i)_{i\geq 0}$ are independent exponential variables of expectation 1 which are independent of the process $(P_i)_{i\geq 0}$. By Theorem 2.3 and some uniform integrability, we infer that

$$\mathbf{E}[i/P_i] \underset{i\to\infty}{\longrightarrow} \mathbf{E}[1/\mathrm{Y}^{\uparrow}(\mathsf{p_q})].$$

The value of the limit is explicit thanks to the theory of positive self-similar Markov processes and equals precisely $2/(\pi^2 \mathsf{p_q})$. We conclude that

$$\frac{1}{\log n}\sum_{i=1}^{n} \mathbf{E}\Big[\frac{1}{2P_i}\Big] \underset{n\to\infty}{\longrightarrow} \frac{1}{\pi^2 \mathsf{p_q}}.$$

We then prove that this sum concentrates around its mean by controlling the correlations to deduce the convergence (2.6).

Remark 2.4. As in Corollary 2.1, we could then compare the balls for the FPP on the dual with those for the graph distance on the original map. Since the latter can be compared with the balls for the dual graph distance, this provides a way to roughly compare metric and FPP balls both in the dual. Again for triangulations much more precise results are due to Curien & Le Gall [CLG19], see also recent extensions to other models [Leh22, Stu22a, AFL22].

2.2.3 Subdiffusivity of the walk

An application of the general result of Gurel-Gurevich & Nachmias [GGN13] shows that the simple random walk on M_∞ for any $a \in (3/2, 5/2]$ is always recurrent [BS14, Ste18]. On the other hand, the random path method and the aforementioned fact that when $a < 2$, the infimum of the FPP-length over all infinite paths in $M_\infty^\dagger$ has finite expectation imply that the random walk on the dual map in this regime is transient [BC17, Corollary 5.2]. In another direction, Benjamini & Curien [BC13] studied the speed of the random walk on quadrangulations and precisely exhibited a *subdiffusive behaviour* (also called *anomalous diffusion*) in the sense that after n steps, the greatest distance to the starting point is at most of order $n^{1/3}$, far from the usual $\sqrt{n}$ for regular lattices. Informally it shows that the map has in itself 'traps', which are zones in which a walker will spend a lot of time, but which do not lead to infinity.

With Nicolas Curien, we first generalised the method of [BC13] in [CM20] by studying the so-called *pioneer points*, which allows to extend the subdiffusive behaviour to the entire dilute regime $a \in (2, 5/2]$, with an upper bound on distances after n steps given by the exponent $1/(2a-2) \in [1/3, 1/2)$. Then in [CM21] we used a completely different method to obtained the upper bound exponent $1/3$ for all $a \in (3/2, 5/2]$. Figure 2.6 recapitulates our results, also (in a weaker sense) for the walk on the dual map.

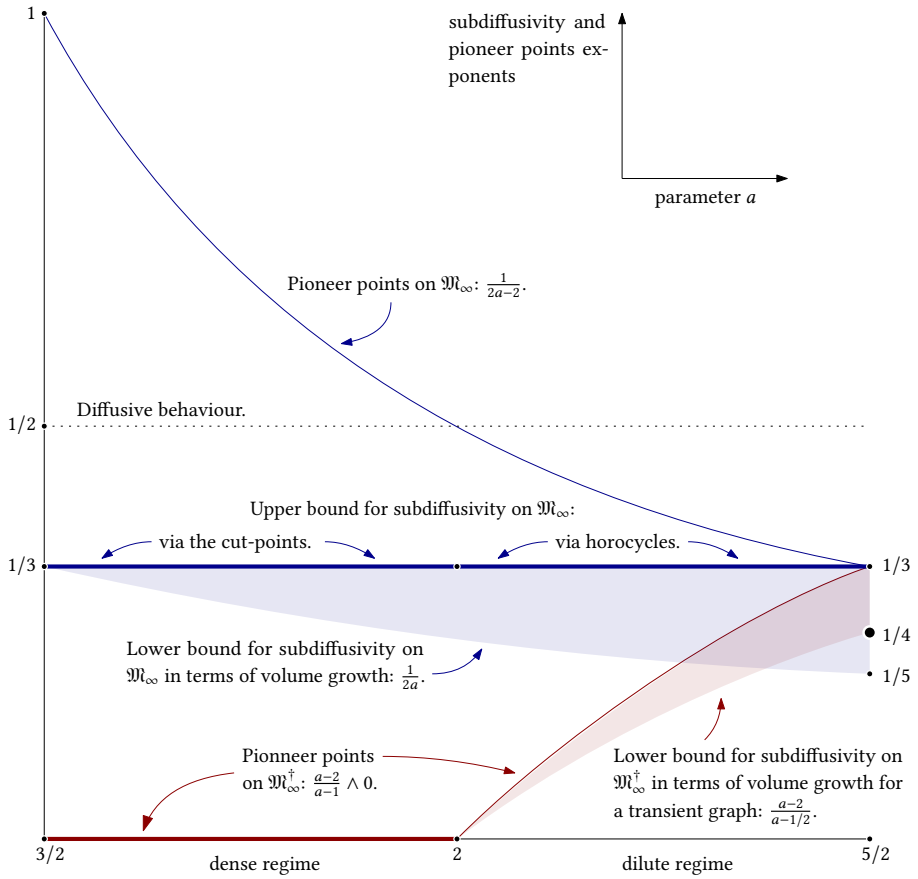


Figure 2.6 – A schematic representation of the bounds on the pioneer points and subdiffusivity exponents for the random walk on M_∞ in blue (top) and the one on $M_\infty^\dagger$ in red (bottom), some of them are still speculative.

Pioneer points of the simple random walk

Let us denote by d_{gr} the graph distance in M_∞. Let us consider $X = (X_n)_{n\geq 0}$, a simple random walk on M_∞ which starts at the root vertex $X_0 = \rho$, and at every step $n \geq 0$, traverses one directed edge originated from X_n chosen uniformly at random.

We say that $\tau \geq 0$ is a *pioneer time* if X_τ lies on the boundary of the unique infinite component when we remove all the faces incident to one of the X_i's for $i < \tau$. If τ is a pioneer time, then X_τ is called a *pioneer point*. By convention $X_0 = \rho$ is a pioneer point, we let $0 = \tau_0 < \tau_1 < \dots$ be the sequence of pioneer times and for every $n \geq 0$, we let $\mathcal{P}_n = X_{\tau_n}$ be the n-th pioneer point.

Corollary 2.2 ([CM20]). *Fix a critical weight sequence* $\mathbf{q}$ *of type* $a \in (\frac{3}{2}, \frac{5}{2}]$. *For every* $\varepsilon > 0$, *there exists* $0 < c_\varepsilon < C_\varepsilon < \infty$ *such that for every* n *large enough,*

$$\mathbf{P}\Big(n^{-\frac{1}{2(a-1)}} \sup_{1\leq k\leq n} d_{\mathrm{gr}}(\rho, \mathcal{P}_k) \in [c_\varepsilon, C_\varepsilon] \Big) \geq 1 - \varepsilon.$$

This follows by applying Theorem 2.5 to the algorithm algorithm $\mathcal{A}_{\mathrm{SRW}}$ which follows the simple random walk. Notice that we explore the map only when the walk reaches a pioneer step, however at such a point, we may trigger more than one peeling step, and as many as needed for the current position of the walk not to lie on the boundary anymore, up to the degree of this point in the whole map. By controlling the degree of the vertices, we stochastically bound this number of steps by a geometric random variable and we prove in [CM20] that the visit of the first n pioneer points triggers a total number of peeling steps which is of order n as well.

This idea can be adapted to study the random walk $X^\dagger = (X_n^\dagger)_{n\geq 0}$ on $M_\infty^\dagger$, started from the root face f_r. Here the peeling exploration given by algorithm $\mathcal{A}_{\mathrm{SRW}^\dagger}$ is simpler: view $X^\dagger$ as going from faces to faces by crossing the edges of M_∞, then either in the next step the walk wants to visit a face already visited, then it does, otherwise, if the edge it is about to cross belongs to the boundary of the explored region, then we peel this edge. We then define the pioneer points $(\mathscr{P}_k^\dagger)_{k\geq 0}$ as the faces of M_∞ in which the walk sits just before crossing the boundary of the explored region. Therefore here the n'th pioneer step is exactly the n'th peeling step. Let us denote by $d_{\mathrm{gr}}^\dagger$ the graph distance in $M_\infty^\dagger$, we directly infer from Theorem 2.5 and Corollary 2.1 the next result.

Corollary 2.3 ([CM20]). *Fix a critical weight sequence* $\mathbf{q}$ *of type* $a \in (\frac{3}{2}, \frac{5}{2}]$*; there exists* $c_a > 0$ *(from* (2.4)*) such that the following holds: For every* $\varepsilon > 0$*, there exist* $0 < c_\varepsilon < C_\varepsilon < \infty$ *such that for every* n *large enough, we have*

$$\begin{aligned}
c_\varepsilon n^{\frac{a-2}{a-1}} &\leq \max_{1\leq k\leq n} d_{\mathrm{gr}}^\dagger(f_r, \mathscr{P}_k^\dagger) \leq C_\varepsilon n^{\frac{a-2}{a-1}} && \text{when} \quad a \in (2, 5/2],\\
(1-\varepsilon)c_a \log n &\leq \max_{1\leq k\leq n} d_{\mathrm{gr}}^\dagger(f_r, \mathscr{P}_k^\dagger) \leq (1+\varepsilon)c_a \log n && \text{when} \quad a \in (3/2, 2),\\
\tfrac{1-\varepsilon}{2\pi^2}\log^2 n &\leq \max_{1\leq k\leq n} d_{\mathrm{gr}}^\dagger(f_r, \mathscr{P}_k^\dagger) \leq \tfrac{1+\varepsilon}{2\pi^2}\log^2 n && \text{when} \quad a = 2.
\end{aligned}$$

with probability at least $1-\varepsilon$.

The random walk is subdiffusive

Let us start with the trivial observation that among the first n steps of a walk, at most n are pioneer steps... Of course the number of pioneer steps may (and probably typically is) much smaller than n since the walk can spend a lot of time in the bulk before reaching the boundary of its past trajectory, but we are not able to control precisely this phenomenon. Still this bound combined with Corollary 2.2 shows that for every $\varepsilon > 0$, there exists $C_\varepsilon \in (0, \infty)$ such that for every n large enough,

$$\mathbf{P}\Big(\sup_{1\leq k\leq n} d_{\mathrm{gr}}(\rho, X_k) \leq C_\varepsilon \cdot n^{\frac{1}{2(a-1)}} \Big) \geq 1-\varepsilon. \tag{2.7}$$

As opposed to the pioneer points, since the walk may visit the interior of the maps which are used to fill in the holes during the exploration here we need Remark 2.3 to replace hulls of balls by balls and thus control the distances. When considering the walk $X^\dagger$ on the dual map, we lack this result and, at least in the dense regime $a < 2$ but this should not be the case when $a > 2$, the hull of the dual ball of radius r could reach distances much larger than r. Therefore we can only deduce from Corollary 2.3 a weaker version, that shows that after n steps, the walk has not left the hull of the ball whose radius is the right-hand side of the equations in this corollary.

Observe that $2(a-1) \in (2, 3]$ for $a \in (2, 5/2]$ so (2.7) shows that the random walk is subdiffusive only in this regime. We improved this result in [CM21].

Theorem 2.8 ([CM21]). *Let* $\mathbf{q}$ *be a critical weight sequence of type* $a \in (\frac{3}{2}, \frac{5}{2}]$*. Under the annealed law of the map together with the random walk, we have*

$$\frac{\sup_{0\leq k\leq n} d_{\mathrm{gr}}(X_0, X_k)}{n^{1/3}\log n} \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

As for the walk on the dual map, if one could replace hulls of balls by balls, then Corollary 2.3 would similarly imply that in the dilute regime $a \in (2, 5/2]$, the walk is subdiffusive, with exponent at most $\frac{a-2}{a-1} \leq \frac{1}{3}$. It also suggests that in the dense regime $a \in (3/2, 2)$, this walk displaces very slowly. Indeed since the volume of the dual balls grow exponentially in the radius [BC17], then we expect the dual distances after n steps of the walk to grow at the speed $\log n$. Our estimates are not tight enough but allow us to prove a $\log^2 n$ upper bound.

Theorem 2.9 ([CM21]). *Let* $\mathbf{q}$ *be a critical weight sequence of type* $a \in (\frac{3}{2}, 2)$*. There exists a constant* $\delta > 0$ *such that under the annealed law of the map together with the random walk, we have*

$$\mathbf{P}\Big(\sup_{0\le k\le n} d_{\mathrm{gr}}^{\dagger}(X_0^{\dagger}, X_k^{\dagger}) \le \delta \log^2 n \Big) \underset{n\to\infty}{\longrightarrow} 1.$$

Let us next explain the strategy to proving Theorem 2.8.

Subdiffusivity from diffusivity on a sparse subgraph We rely on the general idea to upper bound the displacement of a random walk on a graph by 'flashing it' on a certain subgraph, i.e. only keeping its trace on this subgraph. The framework is the following: let $\mathcal{G}$ be a random connected (multi)graph, either finite or infinite, but locally finite, with a distinguished origin vertex ρ, and consider a simple random walk $(X_n)_{n\ge 0}$ on $\mathcal{G}$ started at $X_0 = \rho$. Any finite subset $\mathcal{G}_R$ of vertices of $\mathcal{G}$ is then equipped with a graph structure by declaring that two vertices are adjacent if one can go from one to the other in $\mathcal{G} \setminus \mathcal{G}_R$ (beware, it is not the induced subgraph structure).

Lemma 2.1 ([CM21]). *Let* $(\beta_R)_{R\ge 1}$ *and* $(\gamma_R)_{R\ge 1}$ *be two positive sequences and* $d \ge 1$*. Suppose that for any integer* $R \ge 1$*, we are given a subset of vertices* $\mathcal{G}_R$ *of the graph* $\mathcal{G}$ *such that:*

(i) With high probability as $R \to \infty$*, the ball of radius* R *in* $\mathcal{G}$ *has less than* R^d *vertices;*

(ii) With high probability as $R \to \infty$*, the walk* X *has moved for a distance at least* β_R in $\mathcal{G}_R$ (for the distance in $\mathcal{G}_R$) *before exiting the ball of radius* R *in* $\mathcal{G}$ *(for the distance in* $\mathcal{G}$*);*

(iii) For every $n \ge 1$*, we have that* $\mathbf{P}(X_n \in \mathcal{G}_R) \le \gamma_R^{-1}$*.*

Then with high probability as $R \to \infty$*, the random walker* X_i *stays in the ball of radius* R *in* $\mathcal{G}$ *for every* $i \le \gamma_R \beta_R^2 \log^{-7/4} R$*.*

Let us only sketch the proof and refer to [CM21, Lemma 4] for the details.

- On the one hand, we infer from (iii) that with high probability, the random walk X has spent less than $\beta_R^2 \log^{-3/2} R$ steps in the subset $\mathcal{G}_R$ among the first $\gamma_R \beta_R^2 \log^{-7/4} R$ steps in total.
- On the other hand, very general bounds on reversible Markov chains [LP16, Theorem 13.4] applied to the walk flashed on $\mathcal{G}_R$ allow us to deduce that, on the event where (i) and (ii) are satisfied, with high probability this walker has not reached a point at distance β_R in $\mathcal{G}_R$ in its first $\beta_R^2 \log^{-3/2} R$ steps in this subset.
- Finally by (ii), the random walker needs to move for a distance at least β_R within the graph $\mathcal{G}_R$ in order to escape from the ball of radius R in $\mathcal{G}$ with high probability, so this does not happen in its first $\beta_R^2 \log^{-3/2} R$ steps in this subset and so in its first $\gamma_R \beta_R^2 \log^{-7/4} R$ steps in total.

Heuristic for $\mathcal{G}_R$ The proof of Theorem 2.8 reduces to finding such a subset $\mathcal{G}_R$ which is big enough so β_R is large, but not too big so γ_R is also large. Indeed, a caricature consists in taking $\mathcal{G}_R$ to be the entire ball of radius R, then $\beta_R = R$ but $\gamma_R = 1$, which shows that the walk is at most diffusive; another extreme consists in taking $\mathcal{G}_R$ to be the union of the boundaries of the balls of radius R and $R/2$, which lie at distance 1 in $\mathcal{G}_R$, but now γ_R is quite large and this again would yield a diffusive upper bound. Let us give a heuristic of our choice of $\mathcal{G}_R$, illustrated in Figure 2.7.

A natural guess for $\mathcal{G}_R$ which is thinner than the entire ball $\mathrm{Ball}(M_\infty, R)$ but which still necessitates about R (flashed) steps to traverse it is the set $\widetilde{\mathcal{G}_R}$ of vertices which separates $\mathrm{Ball}(M_\infty, R/2)$ from infinity (see Figure 2.7 left), namely the vertices $u \in M_\infty \setminus \mathrm{Ball}(M_\infty, R/2)$ from which there exists an infinite path along which the distance to ρ is nondecreasing. The main drawback is that estimating $\mathbf{P}(X_n \in \mathcal{G}_R)$ is a difficult task. This is due to the fact that this set strongly depends on ρ. We shall rather construct

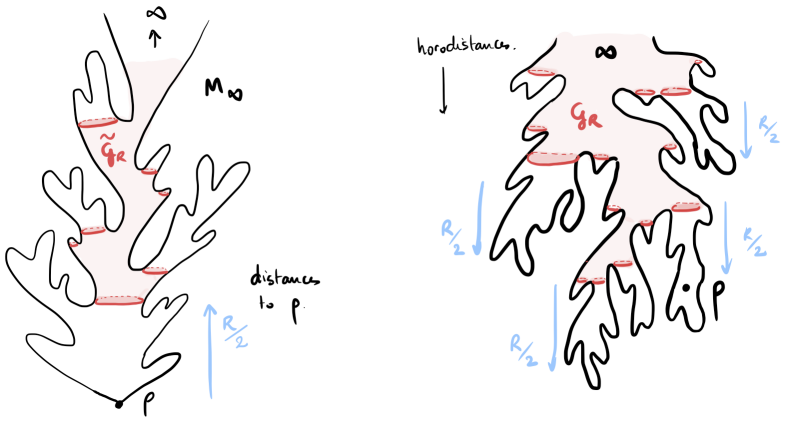


Figure 2.7 – A natural try for $\mathcal{G}_R$ on the left (not stationary) and its stationary version using the (undefined!) horodistances.

our random subsets $\mathcal{G}_R$ in a *stationary* way, i.e. such that $\mathbf{P}(X_n \in \mathcal{G}_R) = \mathbf{P}(\rho \in \mathcal{G}_R)$ for all n. Since the random graph M_∞ is itself stationary [Cur19, Proposition 7.9], then it suffices to construct $\mathcal{G}_R$ in a way that does not depend on the origin ρ of M_∞.

A way to proceed is to use 'distances from infinity' or horodistances rather than distances to ρ. These horodistances are (supposedly) defined by

$$h(u) = \lim_{z \to \infty} d_{\mathrm{gr}}(z, u) - d_{\mathrm{gr}}(z, \rho) \in \mathbf{Z}, \tag{2.8}$$

where $z \to \infty$ means that z escapes from any finite set in the map. Then define the set $\mathcal{G}_R$ as those vertices u such that at least R^{2a-1} (the typical volume of a ball of radius R) different vertices lie 'under u', i.e. may be joined to u by a path visiting only vertices with horodistance $h(u)$ or smaller (see Figure 2.7 right). Since the definition of $\mathcal{G}_R$ does not depend on the origin of the map, it is stationary in the sense that $\mathbf{P}(X_n \in \mathcal{G}_R)$ is constant in n. In the notation of Lemma 2.1 we expect both $\beta_R \approx R$ and $\gamma_R \approx R$, which yields the upper bound of 1/3 on the subdiffusivity exponent by Lemma 2.1.

Actually horodistances (2.8) are not yet proved to exist in general Boltzmann maps, and have only been constructed in the case of quadrangulations and triangulations [CMM13, CM18b]. Therefore, we instead use a trick and emulate them on finite maps: we replace horodistances by the distances to an extra large boundary, far away from the root edge. Indeed, as shown by Budd [Bud17], the infinite Boltzmann map M_∞ also appears as the local limit of finite rooted maps with an extra marked face with perimeter $2\ell \to \infty$. In a sense this large face forces the map to have a large volume but it naturally sits far from the root so it does not affect a finite neighbourhood of the origin.

Formally, instead of $\mathcal{G}_R$ we consider the set $\mathcal{H}_R$ of endpoints of the *R-good edges* defined as follows. Explore the finite map with a marked oriented edge and a face with degree 2ℓ starting from this face and using the algorithm $\mathcal{A}_{\mathrm{metric}}$ but without filling in the holes created during the exploration. Instead continue exploring these holes up to the point where the remaining volume (number of edges) of the map which should fill it in drops below R^{2a-1}. Notice that this exploration is not 'Markovian' since it uses the knowledge of the undiscovered part, but we only use it to define our set of good edges. When the exploration is finished, we get a submap $\mathfrak{e}$ with possibly holes, each of them hiding a map of volume smaller than R^{2a-1}. The set of all edges peeled during this process is the set of R-good edges. See Figure 2.8 right for an illustration. This exploration allows us to show that $\mathcal{H}_R$ fits into the framework of Lemma 2.1 with $\beta_R \approx R$ and $\gamma_R \approx R$, leading to the upper bound of 1/3 in Theorem 2.8.

Subdiffusivity via cut points in the dense phase Fix $a \in (3/2, 2)$; in this regime the map M_∞ possesses *cut edges*, i.e. edges which must be traversed by any path from the origin from infinity; the latter come from large faces which touch themselves. We can derive another proof of Theorem 2.8 in

Figure 2.8 – Illustration of the construction of R-good edges in a map with a boundary: when discovered using the peeling algorithm starting from the large boundary, it remains at least R^{2a-1} edges in the remaining hole to be filled-in ('below').

this regime using these edges [CM21, Section 4.2]. Indeed let $\mathcal{G}_R$ denote the set of all vertices adjacent to edges which separate from infinity a part of the map with volume at least R^{2a-1} (again, the typical volume of the ball of radius R), see Figure 2.9. A cut edge is obtain in a peeling procedure by a first peeling step which discovers a (large) face, and then a second peeling steps that identifies together two edges of this face. The volume separated from infinity is then the increment of the volume at this second step. We rely on the explicit transition probabilities from Theorem 2.2 to prove that $\mathcal{G}_R$ fits into the framework of Lemma 2.1 with $\beta_R \approx R^{4-2a}$ and $\gamma_R \approx R^{4a-5}$. This gives $\gamma_R \beta_R^2 \approx R^3$ and thus the same upper bound of 1/3 on the subdiffusivity exponent! Let us mention that the cut edges yield cut points in the dual map $M_\infty^\dagger$ and are also used to derive Theorem 2.9.

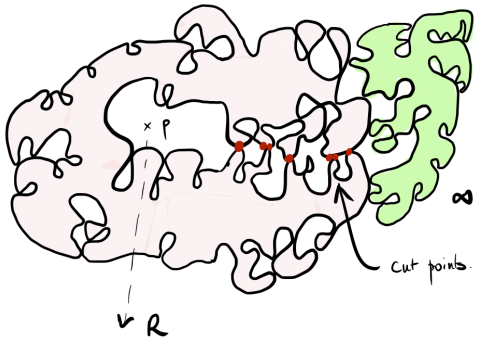


Figure 2.9 – Illustration of the cut points in the dense case. Roughly R^{4-2a} cut points separate the origin from infinity in the ball of radius R. A diffusivity estimate for the random walk flashed on these cut points combined with the density R^{5-4a} of these cut points yields again the upper bound of 1/3 on the subdiffusivity exponent.

Remark 2.5. The control on the number of cut-edges in the ball of radius R also implies that the effective resistance between the origin and the boundary $\partial\overline{\mathrm{Ball}}(M_\infty, R)$ grows at least as some $\log R$ factor (for the vertex degrees) times $R^{4-2a} \longrightarrow \infty$ since $a < 2$. This gives another, quantitative although certainly not optimal, proof of the recurrence of the walk in this regime.

2.2.4 Comments and perspectives

About the random walk

Since we just discussed upper bounds on the subdiffusivity exponent of the walk X on M_∞ and $X^\dagger$on its dual, we can wonder if these are tight, and what is the exact exponent s_a and $s_a^\dagger$ respectively. Figure 2.6 depicts the exponents discussed below.

Original map. Recall that the map M_∞ has polynomial volume growth, with exponent $2a-1$. Then a general result [BC13, Remark p527] suggests that $s_a \geq 1/(2a)$. Note that this lower bound reaches our upper bound $1/3$ when $a \downarrow 3/2$, and gives the interval $[1/5, 1/3]$ when $a = 5/2$. In the case of quadrangulations, it was conjectured by Benjamini & Curien [BC13, Conjecture 1] that the exponent is precisely $1/4$. This was proved for triangulations (of type 2) by Gwynne, Hutchcroft, & Miller [GM21, GH20] who obtained respectively the lower and upper bound for the exponent $1/4$ using a Liouville Quantum Gravity approach. Their powerful method is however specific to this case of triangulations and some closely related models, and the case of quadrangulations is still open. We believe the exponent $1/4$ to hold in the entire regime $a = 5/2$ but we do not have a clear conjecture for the function s_a which would interpolate between $s_{3/2} = 1/3$ and $s_{5/2} = 1/4$.

Dual map. In the case of the dual map, recall that when $a < 2$, we prove in Theorem 2.9 a $\log^2 n$ upper bound for the displacement of the walk $X^\dagger$ after n steps and we believe that the actual speed is $\log n$ given the exponential growth of the dual balls. Similarly when $a = 2$, by Theorem 2.6, the $\log^2 n$ could be correct in this regime; our approach could possibly be adapted to this case, however here we do not expect to have cut points, but rather very short bottlenecks, of logarithmic size. Finally, in the regime $a > 2$, recall from the discussion below (2.7) that the pioneer point approach shows that after n steps, the walk $X^\dagger$ has not left yet the hull of the dual ball with radius $n^{\frac{a-2}{a-1}}$. Controlling the diameter of these hulls, we can expect $s_a^\dagger \leq \frac{a-2}{a-1}$. Recall also from [CLG19] that in the case of triangulations, the distances on the primal and dual maps are asymptotically proportional; we expect thus to have $s_{5/2}^\dagger = s_{5/2} = 1/4$ in the whole $a = 5/2$ regime. Again we do not have a clear conjecture for the function $s_a^\dagger$ which would interpolate between $s_2^\dagger = 0$ and $s_{5/2}^\dagger = 1/4$; we note that the previous upper bound $\frac{a-2}{a-1}$ does.

Transience of the dual walk? We mentioned that the walk X is always recurrent [BS14, Ste18] and that $X^\dagger$ is transient when $a < 2$ [BC17]. The behaviour of $X^\dagger$ when $a \in [2, 5/2]$ is still unknown. Let us point out that the results of [GGN13] do not apply since the vertex degrees in the dual have polynomial tails and not exponential. Still believing that the original map and its dual can be compared when $a = 5/2$, we expect $X^\dagger$ to be recurrent when $a = 5/2$, but possibly transient when $a \in (2, 5/2)$.

More general models

Theorem 2.5 shows that the peeling explorations of any critical stable Boltzmann map are very close to each other. We wonder if this still holds for any critical map, without regularity assumptions on the face degrees. This would require new ideas since Theorem 2.5 strongly relied on volume estimates available thanks to our assumptions. Note that the simple fact that any peeling algorithm always eventually reveals the entire map is not known for all critical maps.

In another direction, Budzinski & Louf [BL22] recently considered local limits of maps with a prescribed face degree sequence, as in Section 1.2, but which are not plane maps, but rather maps on a surface with genus g_n which possibly tends to infinity. They proved the local convergence of this model under natural assumptions on the face degrees and the genus. We wonder if, fixing the genus to be 0 (so it really is the model of Section 1.2), one can obtain a simpler proof, e.g. by comparing this discrete model with a Boltzmann model.

Of course, as in the previous chapter, non bipartite maps remain to be studied for most part...

More on the $a = 2$ regime

Section 2.2.2 and the paper [BCM18] only scratch the surface of the dual maps in the regime $a = 2$. In the dense regime $a < 2$, because of the exponential growth of the dual metric balls, one does not expect any nontrivial scaling limit of such maps conditioned to be large. On the contrary in the dilute regime $a > 2$, which has polynomial growth, one does expect interesting scaling limits, which should be related

to the theory of growth-fragmentation processes [LGR20, BCK18, BBCK18, Ged19, Ged22]. In the case $a = 2$, the intermediate volume growth of the dual balls (Theorem 2.6) suggests again no interesting scaling limit of such maps conditioned to be large. However we believe that if one places a cut-off and restricts to the vertices of the dual map with large degree, then letting first the volume tend to infinity and then the cut-off tend to 0, one can obtain interesting, non-compact, scaling limits which are related to the CLE_4. This is actually the topic of the just starting PhD. thesis of Emmanuel Kammerer, jointly supervised by Nicolas Curien and myself.

2.3 Biconditioned maps, a littler closer

This final section presents a recent work in collaboration with Nicolas Curien and Igor Kortchemski [CKM22]. Let us stress that we consider here general, non necessarily bipartite maps. Also, from a technical point of view, this work has little to do with the rest of this document.

2.3.1 The kernel of sparse plane maps

Let us denote by $M_{n,\mathrm{f}}$ a map sampled uniformly at random with n edges and f faces, and thus $n + 2 - \mathrm{f}$ vertices by Euler's formula. We focus here in the so-called *sparse regime*

$$1 \ll \mathrm{f} \ll n,$$

in which the average vertex degree tends to 2 and face degree to ∞. As opposed to Section 1.3.3, we aim at understanding the geometry of $M_{n,\mathrm{f}}$ in a smaller scale than the diameter, which keeps the faces macroscopic, and see how they are put together.

We rely on a core and kernel decomposition of the maps, which is classical in random graph theory (although the names may vary, such as scheme for kernel) [JKŁP93, Łu91, NRR15, NR18, CMS09, Cha10]. The *core* of a map $\mathfrak{m}$ is the map $\mathrm{Core}(\mathfrak{m})$ obtained from $\mathfrak{m}$ by repeatedly removing vertices of degree 1. Its *kernel* is the map $\mathrm{Ker}(\mathfrak{m})$ obtained from $\mathrm{Core}(\mathfrak{m})$ by replacing all maximal paths of vertices of degree 2 by single edges. When $\mathfrak{m}$ is not a tree, its core and kernel are nonempty. The root edge is canonically transferred from $\mathfrak{m}$ to $\mathrm{Core}(\mathfrak{m})$ and then to $\mathrm{Ker}(\mathfrak{m})$, see Figure 2.10 for an example. Notice that the three maps $\mathfrak{m}$, $\mathrm{Core}(\mathfrak{m})$, and $\mathrm{Ker}(\mathfrak{m})$ all have the same number of faces, which is why we chose it as the second parameter, as opposed to the number of vertices in Section 1.3.3.

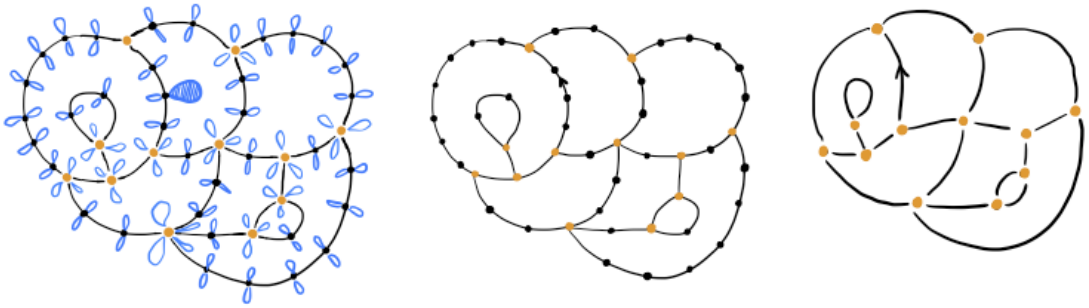


Figure 2.10 – From left to right: a plane map with its root edge in the thick blue tree, its core after removing these subtrees, and its kernel after further contracting vertices of degree two.

By construction, the kernel of a map only has vertices of degree at least 3 and one can wonder if it typically has only vertices of degree exactly 3, in which case it is said to be *trivalent* (or *cubic*). In order to quantify how far from being trivalent such kernel $\mathfrak{K}$ is, we consider its *defect number*, defined by

$$\mathrm{Def}(\mathfrak{K}) = \sum_{v \in \mathrm{Vertices}(\mathfrak{K})} (\deg(v) - 3).$$

Thus a trivalent map has defect 0. Euler's formula relate the defect to the other quantities:

$$\mathrm{Def} = 2\#\mathrm{Edges} - 3\#\mathrm{Vertices} = 3\#\mathrm{Faces} - \#\mathrm{Edges} - 6 = 2\#\mathrm{Faces} - \#\mathrm{Vertices} - 4. \tag{2.9}$$

Therefore, given the number of faces f, trivalent maps are those which maximise the number of edges ($3\mathrm{f}-6$) and of vertices ($2\mathrm{f}-4$), and the defect of a map is the number of edges or vertices less than this maximum (hence the name).

Our first main result [CKM22, Theorem 1] is a phase transition.

Theorem 2.10 ([CKM22]). *The following results hold.*

(i) If $\mathrm{f}_n \ll n^{1/3}$, then the kernel of M_{n,f_n} is trivalent with probability $1 - o(1)$.

(ii) If $\mathrm{f}_n \sim an^{1/3}$, then $\mathrm{Def}(\mathrm{Ker}(M_{n,\mathrm{f}_n}))$ converges in law to the Poisson distribution with mean $\frac{9}{2\sqrt{2}}a^{3/2}$.

(iii) If $n^{1/3} \ll \mathrm{f}_n \ll n$, then

$$\sqrt{\frac{n}{\mathrm{f}_n^3}} \cdot \mathrm{Def}(\mathrm{Ker}(M_{n,\mathrm{f}_n})) \xrightarrow[n\to\infty]{\mathbf{P}} \frac{9}{2\sqrt{2}}.$$

Remark 2.6. A key consequence is that the defect number of the kernel is in any case of order $\sqrt{\mathrm{f}_n^3/n} \ll \mathrm{f}_n$ since $\mathrm{f}_n \ll n$. By (2.9) this shows that the number of edges and of vertices is asymptotically equivalent to that of a trivalent map. We say that in this case that the kernel is *near trivalent*.

The starting point of Theorem 2.10 is the core–kernel decomposition which allows to show that the probability that the kernel of $M_{n,\mathrm{f}}$ is a given candidate $\mathfrak{K}$ with f faces and k edges is proportional to the quantity

$$\Phi_n(k) = \sum_{c=k}^{n} \binom{k}{c}\binom{2n}{n+c}.$$

Indeed, remembering that one should take into account the transfer of the root edge, then the first binomial factor counts the number of possible cores with c edges since they are simply obtained by expanding the k edges of the kernel, whereas the second binomial factor counts the number of possible forests (with the root edge of the map) one can attach on the corners of the core. Consequently, the probability that the kernel of $M_{n,\mathrm{f}}$ has a given defect $\mathrm{d} \geq 0$, or equivalently by (2.9), has $k = 3\mathrm{f} - \mathrm{d} - 6$ edges, is proportional to

$$\Phi_n(3\mathrm{f} - \mathrm{d} - 6) \cdot \#\mathbf{T}(\mathrm{f}, \mathrm{d}), \tag{2.10}$$

where $\mathbf{T}(\mathrm{f}, \mathrm{d})$ denotes the set of all rooted maps with f faces, whose vertices all have degree at least 3, and which have defect number d.

Then the proof of Theorem 2.10 consists roughly in showing that the quantity in (2.10) increases in d up to a value D_n, which is 0 in the first regime and $D_n \sim \frac{9}{2\sqrt{2}}\sqrt{\frac{n}{\mathrm{f}_n^3}}$ in the last one, and it decreases afterwards, and then prove that it concentrates enough around the value for $\mathrm{d} = D_n$ to ensure the convergence. Thanks to the explicit formula for Φ_n, this term can be controlled by brute calculation. For $\#\mathbf{T}(\mathrm{f}, \mathrm{d})$ we design a *contraction operation* which basically consists in viewing such a map with defect d as being obtained from a trivalent map by contracting d edges of the latter, see Figure 2.11 for an example. Indeed contracting an edge between two vertices of degree 3 replaces them by a single vertex of degree 4, and thus increases the defect by one. However such a contracted edge must be a non loop edge. More generally, the subset of d edges must contain no cycle, so this induces a bias and a uniform random map in $\mathbf{T}(\mathrm{f}, \mathrm{d})$ is not constructed from a uniform random trivalent map. By relying on a recent work of Budzinski [Bud21, Theorem 2] who obtained large deviation estimates for the proportion of non loop edges (or more general patterns) in trivalent maps, it can be shown that a uniform random map with f faces and $\mathrm{d} = o(\mathrm{f})$ defects (which is the regime we consider in Theorem 2.10, recall Remark 2.6) is close to the map obtained from a uniform random trivalent map with f faces by contracting d non loop edges sampled uniform random. This allows to study the asymptotic behaviour of $\#\mathbf{T}(\mathrm{f}, \mathrm{d})$.

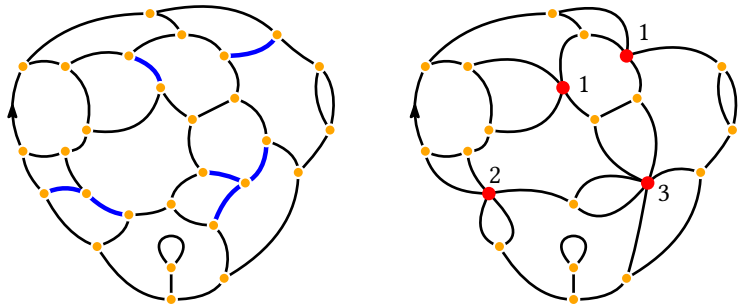


Figure 2.11 – *Left:* a trivalent map with a subset of d = 7 edges in blue containing no cycle. *Right:* the trivalent map with d defects obtained by contracting these blue edges; the positive defects are indicated next to vertices with degree more than 3.

As an interesting corollary, we obtain an explicit asymptotic estimate for the number of sparse plane maps. Indeed, the asymptotic number of maps with n edges and $\mathrm{f}_n \sim cn$ faces with $c \in (0, \infty)$ is given in [BCR93, Theorem 1], we here consider the regime $\mathrm{f}_n = O(n^{1/3})$.

Corollary 2.4. *The number of plane maps with n edges and* $\mathrm{f}_n \sim cn^{1/3}$ *faces with* $c \in [0, \infty)$ *is asymptotically equivalent to*

$$\frac{\exp(-(2-\sqrt{3})(3c/2)^{3/2})}{4\pi} \cdot n^{-3} \cdot 4^n \cdot \left(2^{1/3}\mathrm{e} \cdot \frac{n}{\mathrm{f}_n}\right)^{3\mathrm{f}_n/2}.$$

2.3.2 Geometry at a mesoscopic scale

Local limit of the kernel Recall that our motivation was to understand the geometry of M_{n,f_n} at a scale in which the faces are macroscopic. Let us start again from its kernel. By Theorem 2.10, when $\mathrm{f}_n \ll n^{1/3}$, the kernel is trivalent with high probability. In this case it has the uniform distribution on trivalent maps, i.e. it is the dual of a uniform random triangulation (of type 1), so by Stephenson [Ste18] it converges in distribution for the local topology to the dual of the UIPT (recall that Angel & Schramm [AS03] considered type 2 or 3 triangulations). This result has recently been extended to 'near triangulations' by Budzinski [Bud21]. Precisely, passing to the dual again (denoted below with a † symbol), Corollary 2 in [Bud21] and Remark 2.6 here show that as soon as $\mathrm{f}_n \ll n$, we have

$$\mathrm{Ker}(M_{n,\mathrm{f}_n}) \xrightarrow[n\to\infty]{(d)} \mathrm{UIPT}^{\dagger},$$

for the local topology.

Semi-continuous limit of sparse maps Recall that the map M_{n,f_n} may be recovered from its kernel by first replacing each edge of the latter by a chain of vertices to reconstruct the core, and then attaching a rooted plane tree in each corner of the core. This can be achieved in a single step by replacing each edge of the kernel by a non rooted plane tree with two marked vertices, which correspond to the two extremities of the edge of the kernel, as shown in Figure 2.12. The tree that replaces the root edge of the kernel (hereafter called the 'root tree') carries itself an oriented edge which is the root edge of the map and which induces a bias.

Then from the combinatorial description mentioned above, one can express the law of theses bipointed trees and show that, apart from the root tree, they converge in the product Gromov–Hausdorff topology at the scale $\sqrt{n/\mathrm{f}_n}$ to i.i.d. Brownian CRT's with two marked points $(\mathcal{T}_h, a, b)$ defined similarly as in (1.2), except that the function h is not the Brownian excursion with duration 1. Instead, h has the law of a size-biased excursion of a Brownian motion with negative drift $\sqrt{3/2}$, and then a is the image of 0 in $\mathcal{T}_h$ and b that of a uniform random time in the excursion. By Girsanov's formula, the Itō excursion measure of this drifted Brownian motion reduces to that of the non drifted one with an exponential

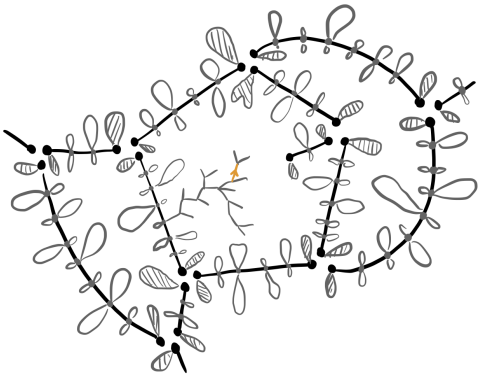


Figure 2.12 – Bipointed trees replacing the edges of the kernel to reconstruct the whole map.

size-bias, i.e. the volume of the tree is biased, but its geometry is still that of the standard CRT. As for the root tree, the limit is a twice-size biased version of such an excursion, with a third uniform marked point ρ which will play the role of the origin of the root edge of the map.

Define thus a locally compact metric space $\mathfrak{F}_{\mathrm{Plan}}$ by taking the dual of the UIPT and replacing each edge by these bipointed CRT's and by the threepointed one for the root edge, which thus carries a marked vertex ρ. As for the topology, we say that a sequence of locally compact metric spaces with a marked point converges in the *local pointed Gromov–Hausdorff topology* when for every positive real $r > 0$, the ball of radius r centred at the marked point converges to that of the limit space for the Gromov–Hausdorff topology. We then arrive at the following convergence result.

Theorem 2.11 ([CKM22]). *Suppose that* $1 \ll \mathrm{f}_n \ll n$*, then the convergence in distribution*

$$\sqrt{\frac{\mathrm{f}_n}{n}} \cdot M_{n,\mathrm{f}_n} \xrightarrow[n\to\infty]{(d)} \mathfrak{F}_{\mathrm{Plan}},$$

holds in the local pointed Gromov–Hausdorff topology.

2.3.3 Unicellular maps in high genus

Let us note that multiple intermediate steps do not use the planarity of the maps. Indeed sample a map uniformly at random with n edges, f_n faces, and genus g_n. By Euler's formula it has now $n - \mathrm{f}_n - 2\mathrm{g}_n + 2$ vertices and the sparse regime is given by

$$1 \ll \mathrm{f}_n + 2\mathrm{g}_n \ll n.$$

The core–kernel decomposition still applies as previously, and the conditional law of the map given the kernel is exactly the same as previously. In particular the law of the defect number of the kernel is still given by (2.10), with the same function Φ_n as previously, only the number of maps with a given defect is different. Finally the contraction operation still applies in a similar way.

The two steps where we used the planarity are the two following, related to [Bud21]:

- To control the number of loops in a trivalent map to understand the contraction operation.
- To get the local convergence of the kernel.

Besides this, the convergence of the maps follows as previously, where f_n should be replaced by $\mathrm{f}_n + 2\mathrm{g}_n$ in the scaling factors. In particular, suppose that we are able to prove that the kernel converges in distribution for the local topology to an infinite random map K_∞. Then the map, now rescaled by $\sqrt{(\mathrm{f}_n + 2\mathrm{g}_n)/n}$, converges in distribution for the local pointed GH topology to the space obtained by replacing each edge of K_∞ by exactly the same pointed Brownian CRT's as above.

In [CKM22] we provide the convergence of the kernel in the extreme case of maps with a single face. These are actively studied [ACCR13, Ray15, Lou21, JL21], in part because they are simple models related to hyperbolic geometry. Indeed, local limits of high genus maps are often still planar, roughly speaking because one cannot see the end of a handle by looking only in a finite neighbourhood of the root, but these handles do provide a negative curvature. In the case of a unicellular map, we thus obtain in the limit a plane map with one face, i.e. a tree [ACCR13]. Recall that we are interested in the kernel of the maps, which are trivalent up to some defect. If, as previously, the defect is small, then we can expect the limit to moreover be trivalent. For maps with a single face, we indeed prove in [CKM22, Equation 24] that the limit of the kernel when $\mathrm{f}_n = 1$ and $\mathrm{g}_n \ll n$ is the infinite three-regular tree. By the above reasoning, the limit of the whole map, rescaled by $\sqrt{2\mathrm{g}_n/n}$, is this three-regular tree in which each edge is replaced by pointed Brownian CRT's.

2.3.4 Comments and perspectives

Non unicellular maps To continue the previous discussion, one can consider the whole regime $\mathrm{f}_n + 2\mathrm{g}_n \ll n$. One can expect the case $\mathrm{g}_n \ll \mathrm{f}_n \ll n$ to be close to the planar case and $\mathrm{f}_n \ll \mathrm{g}_n \ll n$ close to the unicellular one. On the other hand, when f_n and g_n are comparable, then for purely trivalent maps, by Budzinski & Louf [BL21] the maps converge to the dual of a Planar Stochastic Infinite Triangulation of [Cur16]. We believe that in the sparse regime, the kernel is close to be trivalent and thus converges to this limit as well. Therefore the mesoscopic scaling limits of the maps should be given by replacing each edge of these objects by the same bipointed CRT's as above. This is however more involved than the two extreme regimes studied so far in [CKM22].

On the global geometry In the unicellular case in high genus, also in the sparse regime $\mathrm{g}_n \ll n$, Janson & Louf [JL21] very recently studied the statistics of the lengths of short cycles, which strikingly matches those of the lengths of short non-contractible curves in Weil–Petersson random surfaces [MP19] in hyperbolic geometry. Note that the cycles are entirely determined by the core; we believe the results of [JL21] could be recovered by considering the kernel of the maps and a proof of this could also give access to other global quantities of sparse and near trivalent unicellular maps such as their diameter. The behaviour of the diameter of high genus maps, which is expected to be logarithmic, is only known in the unicellular case [Ray15] with no further constraint, but not even in simple cases such as quadrangulations [Lou22, Conjecture 1].

Other scales in the planar case In addition to Theorem 2.11 on the convergence of the map M_{n,f_n} at the scale $\sqrt{n/\mathrm{f}_n}$, we also consider smaller scalings in [CKM22, Proposition 17]. Roughly speaking, $\sqrt{n/\mathrm{f}_n}$ is the scale of the distance from the origin of the map to its core (i.e. the diameter of the blue tree in Figure 2.10 left), so under a smaller scaling, the neighbourhood of the root is tree-like. Precisely, without any rescaling, the map converges locally to the uniform infinite discrete tree (alternatively a critical geometric Bienaymé–Galton–Watson tree conditioned to survive), whereas under any scaling $1 \ll s_n \ll \sqrt{n/\mathrm{f}_n}$ it converges for the local pointed Gromov–Hausdorff topology to the Self-Similar Continuum Random Tree [Ald91b, Section 2.5] coded by a two-sided Brownian motion. One can then turn to the behaviour of the map at a scaling $s_n \gg \sqrt{n/\mathrm{f}_n}$. According to Curien & Le Gall [CLG19], uniform random trivalent plane maps with f_n faces converge at the scaling $\mathrm{f}_n^{1/4}$ to the Brownian sphere. We expect more generally the kernel of M_{n,f_n} to do so when $\mathrm{f}_n \ll n$. Our previous argument then indicates that the diameter of M_{n,f_n} grows like $\mathrm{f}_n^{1/4}\sqrt{n/\mathrm{f}_n} = (n^2/\mathrm{f}_n)^{1/4}$ and we expect again the Brownian sphere in the limit at this scaling, see precisely [CKM22, Conjecture 18]. Finally at a scale $\sqrt{n/\mathrm{f}_n} \ll s_n \ll (n^2/\mathrm{f}_n)^{1/4}$, we expect the map to converge now to the Brownian *plane* [CLG14].

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