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Simply Generated Non-Crossing Partitions

IGOR KORTCHEMSKI[1†] and CYRIL MARZOUK[2‡]

[1] CNRS, CMAP, École Polytechnique, Université Paris-Saclay, Route de Saclay, 91128 Palaiseau CEDEX, France
(e-mail: igor.kortchemski@normalesup.org)

[2] Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
(e-mail: cyril.marzouk@math.uzh.ch)

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We introduce and study the model of simply generated non-crossing partitions, which are, roughly speaking, chosen at random according to a sequence of weights. This framework encompasses the particular case of uniform non-crossing partitions with constraints on their block sizes. Our main tool is a bijection between non-crossing partitions and plane trees, which maps such simply generated non-crossing partitions into simply generated trees so that blocks of size k are in correspondence with vertices of out-degree k. This allows us to obtain limit theorems concerning the block structure of simply generated non-crossing partitions. We apply our results in free probability by giving a simple formula relating the maximum of the support of a compactly supported probability measure on the real line in terms of its free cumulants.

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1. Introduction

We are interested in the structure of non-crossing partitions. The latter were introduced by Kreweras [28], and quickly became a standard object in combinatorics. They have also appeared in many other different contexts, such as low-dimensional topology, geometric group theory and free probability (see *e.g.* the survey [32] and the references therein). In this work, we study combinatorial and geometric aspects of large *random* non-crossing partitions.

Recall that a partition of $[n] \coloneqq \{1, 2, \ldots, n\}$ is a collection of (pairwise) disjoint subsets, called *blocks*, whose union is $[n]$. A *non-crossing partition* of $[n]$ is a partition of the

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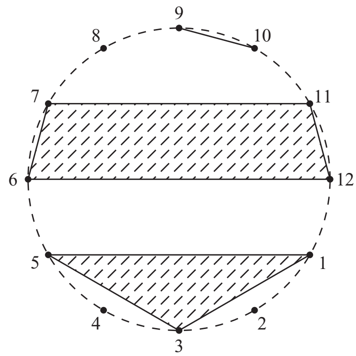


Figure 1. The non-crossing partition $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ of [12].

vertices of a regular n-gon (labelled by the set $[n]$ in clockwise order) with the property that the convex hulls of its blocks are pairwise disjoint (see Figure 1 for an example).

Large discrete combinatorial structures. There are many ways to study discrete structures. Given a finite combinatorial class $\mathcal{A}_n$ of objects of 'size' n, a first step is often to calculate as explicitly as possible its cardinal $\#\mathcal{A}_n$, using for instance bijective arguments or generating functions. For non-crossing partitions, it is well known that they are enumerated by Catalan numbers. It is also often of interest to enumerate elements of $\mathcal{A}_n$ satisfying constraints. For instance, the number of non-crossing partitions of $[n]$ with given block sizes [28] or the total number of blocks [19] have been studied. Edelman [19] also introduced and enumerated k-*divisible* non-crossing partitions (where all blocks must have size divisible by k), which have also been studied by Arizmendi and Vargas [4] in connection with free probability. Arizmendi and Vargas also studied k-*equal* non-crossing partitions (where all blocks must have size exactly k).

In probabilistic combinatorics, one is interested in the properties of a *typical* element of $\mathcal{A}_n$. In other words, one studies statistics of a random element a_n of $\mathcal{A}_n$ chosen uniformly at random. Graph-theoretic properties of different uniform plane non-crossing structures obtained from a regular polygon have been considered in recent years. For example, [16, 21, 17, 12] study the maximal degree in random triangulations, [7, 12] obtain concentration bounds for the maximal degree in random dissections, and [31, 15, 12] are interested in the structure of non-crossing trees. However, uniform non-crossing partitions have attracted less attention. Arizmendi [3] finds the expected number of blocks of given size for non-crossing partitions of $[n]$ with certain constraints on the block sizes, Ortmann [34] shows that the distribution of a uniform random block in a uniform non-crossing partition P_n of $[n]$ converges to a geometric random variable of parameter $1/2$ as $n \to \infty$, and Curien and Kortchemski [12] have obtained limit theorems concerning the length of the longest chord of P_n.

It is also of interest to sample an element a_n of $\mathcal{A}_n$ according to a probability distribution different from the uniform law; one then studies the impact of this change on the asymptotic behaviour of a_n as $n \to \infty$. Certain families of probability distributions lead to the same asymptotic properties, and are said to belong to the same universality class.

However, the structure of a_n may be impacted drastically. To the best of our knowledge, only uniform non-crossing partitions have yet been studied; see [4, 12, 34].

Finally, another direction is to study distributional limits of a_n. Indeed, if it is possible to see the elements of the combinatorial class under consideration as elements of the same metric space, it makes sense to study the convergence in distribution of the sequence of random variables $(\mathsf{a}_n)_{n\geqslant 1}$ in this metric space. In the case of uniform non-crossing partitions, this approach has been followed in [12] by seeing them as compact subsets of the unit disk; we extend the result obtained there to simply generated non-crossing partitions.

Simply generated non-crossing partitions. In this work, we propose to sample non-crossing partitions at random according to a Boltzmann-type distribution, which depends on a sequence of weights. For every integer $n \geqslant 1$, let $\mathbb{NC}_n$ denote the set of all non-crossing partitions of $[n]$. Given a sequence of non-negative real numbers $w = (w(i); i \geqslant 1)$, with every partition $P \in \mathbb{NC}_n$, we associate a weight $\Omega^w(P)$:

$$\Omega^w(P) = \prod_{B \text{ block of } P} w(\text{size of } B).$$

Implicitly, we shall always restrict our attention to those values of n for which $\sum_{P\in\mathbb{NC}_n} \Omega^w(P) > 0$. Then, for every $P \in \mathbb{NC}_n$, set

$$\mathbb{P}_n^w(P) = \frac{\Omega^w(P)}{\sum_{Q\in\mathbb{NC}_n} \Omega^w(Q)}.$$

A random non-crossing partition of $[n]$ sampled according to $\mathbb{P}_n^w$ is called a *simply generated non-crossing partition*. We chose this terminology because of the similarity to the model of simply generated trees, introduced by Meir and Moon [33], and whose definition we recall in Section 2.2 below. We were also inspired by recent work on scaling limits of Boltzmann-type random graphs [27, 29].

We point out that, taking $w(i) = 1$ for every $i \geqslant 1$, $\mathbb{P}_n^w$ is the uniform distribution on $\mathbb{NC}_n$; more generally, if $\mathcal{A}$ is a non-empty subset of $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $w_{\mathcal{A}}(i) = 1$ if $i \in \mathcal{A}$ and $w_{\mathcal{A}}(i) = 0$ if $i \notin \mathcal{A}$, then $\mathbb{P}_n^{w_{\mathcal{A}}}$ is the uniform distribution on the subset of $\mathbb{NC}_n$ formed by partitions with all block sizes belonging to $\mathcal{A}$ (provided that they exist), and which we call *$\mathcal{A}$-constrained non-crossing partitions* (see Fig. 3 for an example). In particular, by taking $\mathcal{A} = \{k\}$ one gets uniform k-equal non-crossing partitions, and by taking $\mathcal{A} = k\mathbb{N}$ one gets uniform k-divisible non-crossing partitions.

Bijections between non-crossing partitions and plane trees. Our main tools for studying simply generated non-crossing partitions are bijections with plane trees. Here we explain the main ideas, and we refer to Section 2.1 for the details. With a non-crossing partition, we start by associating a (two-type) dual tree, as depicted in Figure 2.

We choose an appropriate root for this two-type tree, and then apply a recent bijection due to Janson and Stefánsson [26]; this yields a bijection $\mathcal{B}^\circ$ between $\mathbb{NC}_n$ and plane trees with $n + 1$ vertices. We mention here that this bijection was directly defined by Dershowitz and Zaks [14] without using the dual two-type tree. It turns out that other known bijections between non-crossing partitions and plane trees, such as Prodinger's bijection [36] and

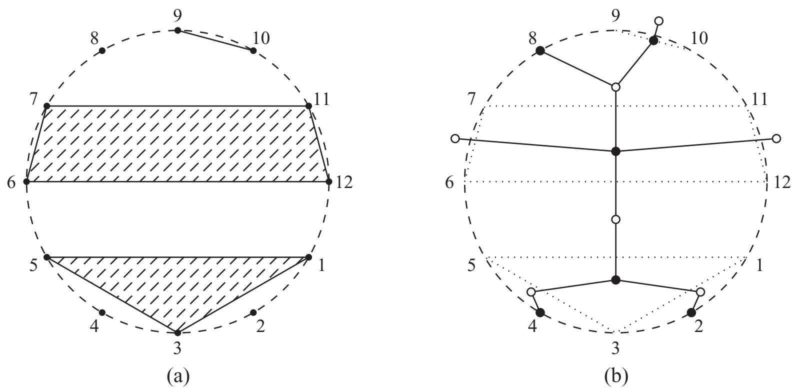


Figure 2. (a) The (non-crossing) partition $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ and (b) its dual tree.

the Kreweras complement [28], can be obtained by choosing to distinguish another root in the dual two-type tree (see again Section 2.1 below for details). Our contribution is therefore to unify previously known bijections between non-crossing partitions and plane trees by showing that they all amount to doing certain operations on the dual tree of a non-crossing partition, and to use them to study *random* non-crossing partitions.

It turns out that the dual tree of a simply generated non-crossing partition is a two-type simply generated tree (Proposition 2.4). A crucial feature of the bijection $\mathcal{B}^\circ$ is that it maps simply generated non-crossing partitions into simply generated trees in such a way that blocks of size k are in correspondence with vertices with out-degree k (Proposition 2.3). This allows us to reformulate questions on simply generated non-crossing partitions involving block sizes in terms of simply generated trees involving out-degrees. The point is that the study of simply generated trees is a well-paved road. In particular, this allows us to show that if P_n is a simply generated non-crossing plane partition of $[n]$, then, under certain conditions, the size of a block chosen uniformly at random in P_n converges in distribution as $n \to \infty$ to an explicit probability distribution depending on the weights. We also obtain, for a certain family of weights, asymptotic normality of the block sizes and limit theorems for the sizes of the largest blocks. We specify here some of these results for $\mathcal{A}$-constrained non-crossing partitions, and refer to Section 3.4 for more general statements and further applications.

Theorem 1.1. *Let $\mathcal{A}$ be a non-empty subset of $\mathbb{N}$ with $\mathcal{A} \neq \{1\}$, and let $P_n^{\mathcal{A}}$ be a random non-crossing partition chosen uniformly at random among all those with block sizes belonging to $\mathcal{A}$ (provided that they exist). Let $\pi_{\mathcal{A}}$ be the probability measure on $\mathbb{Z}_+ = \{0,1,2,\dots\}$ defined by*

$$\pi_{\mathcal{A}}(k) = \frac{\xi_{\mathcal{A}}^k}{1+\sum_{i\in\mathcal{A}} \xi_{\mathcal{A}}^i} \mathbb{1}_{k\in\{0\}\cup\mathcal{A}}, \quad \text{where } \xi_{\mathcal{A}} > 0 \text{ is such that } 1 + \sum_{i\in\mathcal{A}} \xi_{\mathcal{A}}^i = \sum_{i\in\mathcal{A}} i \cdot \xi_{\mathcal{A}}^i.$$

(i) *Let $S_1(P_n^{\mathcal{A}})$ be the size of the block containing 1 in $P_n^{\mathcal{A}}$. Then, for every $k \geqslant 1$,*

$$\mathbb{P}(S_1(P_n^{\mathcal{A}}) = k) \to k\pi_{\mathcal{A}}(k) \quad \text{as } n \to \infty.$$

(ii) *Let B_n be a block chosen uniformly at random in $P_n^{\mathcal{A}}$. Then, for every $k \geqslant 1$,*

$$\mathbb{P}(|B_n| = k) \to \pi_{\mathcal{A}}(k)/(1 - \pi_{\mathcal{A}}(0)) \quad \text{as } n \to \infty.$$

(iii) *Let C be a subset of $\mathbb{N}$ and let $\zeta_C(P_n^{\mathcal{A}})$ denote the number of blocks of $P_n^{\mathcal{A}}$ whose size belongs to C. As $n \to \infty$, the convergence $\zeta_C(P_n^{\mathcal{A}})/n \to \pi_{\mathcal{A}}(C)$ holds in probability and, in addition,*

$$\mathbb{E}[\zeta_C(P_n^{\mathcal{A}})]/n \to \pi_{\mathcal{A}}(C).$$

In the particular case of uniform k-divisible non-crossing partitions, Theorem 1.1(ii,iii) has been obtained by Ortmann [34, Section 2.3]. Also, Arizmendi [3] obtained by combinatorial means closed formulas for the expected number of blocks of given size in k-divisible non-crossing partitions.

Applications in free probability. An additional motivation for introducing simply generated non-crossing partitions comes from free probability. Indeed, the *partition function*

$$Z_n^w := \sum_{P \in \mathbb{NC}_n} \prod_{B \text{ block of } P} w(\text{size of } B)$$

expresses the moments of a measure in terms of its free cumulants. More precisely, if μ is a probability measure on $\mathbb{R}$ with compact support, its Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(\mathrm{d}t)}{z - t}, \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu$$

is analytic and locally invertible on a neighbourhood of ∞; its inverse K_μ is meromorphic around zero, with a simple pole of residue 1 (see *e.g.* [6, Section 5]). One can then write

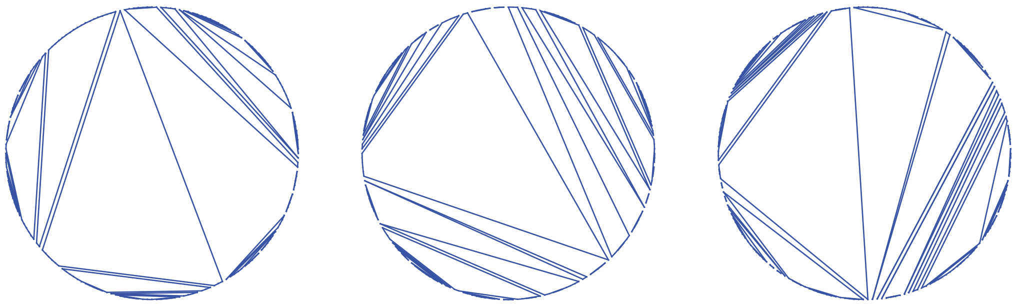
$$R_\mu(z) = K_\mu(z) - \frac{1}{z} = \sum_{n=0}^{\infty} \kappa_{n+1}(\mu) z^n.$$

The analytic function R_μ is called the *R-transform of* μ, and uniquely defines μ. In addition, the coefficients $(\kappa_n(\mu); n \geqslant 1)$ are called the *free cumulants* of μ. The importance of R-transforms stems from the fact that they linearize free additive convolution and characterize weak convergence of probability measures; see [6]. The following relation between the moments of μ and its free cumulants is a well-known fact, that goes back to [38]. Let μ be a compactly supported probability measure on $\mathbb{R}$. Then, for every $n \geqslant 1$,

$$\int_{\mathbb{R}} t^n \mu(\mathrm{d}t) = \sum_{P \in \mathbb{NC}_n} \prod_{B \text{ block of } P} \kappa_{\text{size}(B)}(\mu). \tag{1.1}$$

In other words, the nth moment of μ is the partition function of simply generated non-crossing partitions on $[n]$ with weights $w(i) = \kappa_i(\mu)$ given by the free cumulants of μ. Using the bijection $\mathcal{B}^\circ$, we establish the following result.

Theorem 1.2. *Let μ be a compactly supported probability measure on $\mathbb{R}$, different from a Dirac mass, and such that all its free cumulants $(\kappa_i(\mu); i \geqslant 1)$ are non-negative. Let s_μ be*



(a) (b) (c)

Figure 3. Simulations of random non-crossing partitions of [200] chosen uniformly at random among all those having only block sizes that are multiples of (a) 5, (b) odd and (c) prime numbers.

the maximum of its support. Set

$$\rho = \left(\limsup_{n\to\infty} \kappa_n(\mu)^{1/n}\right)^{-1} \quad \textit{and} \quad \nu = 1 + \lim_{t\uparrow\rho} \frac{t^2 R'_\mu(t) - 1}{tR_\mu(t) + 1}.$$

If $\nu \geqslant 1$*, there exists a unique number* ξ *in* $(0, \rho]$ *such that* $R'_\mu(\xi) = 1/\xi^2$*, and, in addition,*

$$s_\mu = \begin{cases} \dfrac{1}{\xi} + R_\mu(\xi) & \textit{if } \nu \geqslant 1, \\ \dfrac{1}{\rho} + R_\mu(\rho) & \textit{if } \nu < 1. \end{cases}$$

See Section 3.3 for examples. This gives a more explicit formula than that obtained by Ortmann [34, Theorem 5.4], which reads

$$\log(s_\mu) = \sup\left\{ \frac{1}{m_1(p)} \sum_{n\in L} p_n \log\left(\frac{\kappa_n(\mu)}{p_n}\right) - \frac{\theta(m_1(p))}{m_1(p)};\ p \in \mathfrak{M}_1^1(L) \right\},$$

where $L = \{n \geqslant 1; \kappa_n(\mu) \neq 0\}$, $\theta(x) = \log(x-1) - x\log(x - 1/x)$, $\mathfrak{M}_1^1(L)$ is the set of probability measures $p = (p_n; n \in \mathbb{N})$ on $\mathbb{N}$ with $p(L^c) = 0$ and $m_1(p)$ is the mean of p.

Non-crossing partitions seen as compact subsets of the unit disk. Finally, if P_n is a simply generated non-crossing partition of $[n]$, we study the distributional limits of P_n, seen as a compact subset of the unit disk by identifying each integer $l \in [n]$ with the complex number $e^{-2i\pi l/n}$. This route was followed in [12], where it was shown that as $n \to \infty$, a uniform non-crossing partition of $[n]$ converges in distribution to Aldous's *Brownian triangulation* of the disk [2], in the space of all compact subsets of the unit disk equipped with the Hausdorff metric, and where the Brownian triangulation is a random compact subset of the unit disk constructed from the Brownian excursion. We show more generally that a whole family of simply generated non-crossing partitions of $[n]$ (including all uniform $\mathcal{A}$-constrained non-crossing partitions) converge in distribution to the Brownian triangulation, and show that other families converge in distribution to a *stable lamination*, which is another random compact subset of the unit disk introduced in [27]. We refer to Section 4 for details and precise statements.

In particular, this has applications concerning the length of the longest chord of P_n. By definition, the *(angular) length* of a chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ with $0 \leqslant s \leqslant t \leqslant 1$ is $\min(t-s, 1-t+s)$. Let $\mathsf{C}(P_n)$ denote the length of the longest chord of P_n. In the case of $\mathcal{A}$-constrained non-crossing partitions, we prove in particular the following result.

Theorem 1.3. *Let $\mathcal{A}$ be a non-empty subset of $\mathbb{N}$ with $\mathcal{A} \neq \{1\}$ and let $P_n^{\mathcal{A}}$ be a random non-crossing partition chosen uniformly at random among all those with block sizes belonging to $\mathcal{A}$ (provided that they exist). Then, as $n \to \infty$, $\mathsf{C}(P_n^{\mathcal{A}})$ converges in distribution to a random variable with distribution*

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbb{1}_{1/3 \leqslant x \leqslant 1/2}\mathrm{d}x.$$

It is remarkable that the limiting distribution in Theorem 1.3 does not depend on $\mathcal{A}$ (it seems that this is *not* the case for the largest block area: see Section 5).

This bears some similarity to [12], but we emphasize that this is not a simple adaptation of the arguments of [12]. Indeed, roughly speaking, [12] manages to code uniform non-crossing partitions of $[n]$ by a dual-type uniform plane tree. In the more general case of simply generated non-crossing partitions, the dual tree is a more complicated two-type tree and the Janson–Stefánsson bijection is needed.

2. Bijections between non-crossing partitions and plane trees

We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk of the complex plane, by $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle, and by $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ the closed unit disk. For every $x, y \in \mathbb{S}^1$, we write $[x, y]$ for the line segment between x and y in $\overline{\mathbb{D}}$, with the convention $[x, x] = \{x\}$. A *geodesic lamination* L of $\overline{\mathbb{D}}$ is a closed subset of $\overline{\mathbb{D}}$ which can be written as the union of a collection of non-crossing such chords, that is, which do not intersect in $\mathbb{D}$. In this paper, by *lamination* we will always mean geodesic lamination of $\overline{\mathbb{D}}$.

We view a partition of $[n]$ as a closed subset of $\overline{\mathbb{D}}$ by identifying each integer $l \in [n]$ with the complex number $\mathrm{e}^{-2\mathrm{i}\pi l/n}$ and by drawing a chord $[\mathrm{e}^{-2\mathrm{i}\pi l/n}, \mathrm{e}^{-2\mathrm{i}\pi l'/n}]$ whenever $l, l' \in [n]$ are two consecutive elements of the same block of the partition, where the smallest and the largest element of a block are consecutive by convention. The partition is non-crossing if and only if these chords do not cross; we implicitly identify a non-crossing partition with the associated lamination throughout this paper.

Let $\mathbb{T}$ be the set of all finite plane trees (see the definition below), and let $\mathbb{T}_n$ be the set of all plane trees with n vertices. We construct two bijections between $\mathbb{NC}_n$ and $\mathbb{T}_{n+1}$. The study of a (random) non-crossing partition then reduces to that of the associated (random) plane tree.

2.1. Non-crossing partitions and plane trees

Recall that $\mathbb{N} = \{1, 2, \dots\}$ is the set of all positive integers, set $\mathbb{N}^0 = \{\varnothing\}$, and let

$$\mathcal{U} = \bigcup_{n \geqslant 0} \mathbb{N}^n.$$

For $u = (u_1, \dots, u_n) \in \mathcal{U}$, we let $|u| = n$ denote the length of u. If $n \geqslant 1$, we define $pr(u) = (u_1, \dots, u_{n-1})$ and for $i \geqslant 1$, we let $ui = (u_1, \dots, u_n, i)$. More generally, for $v = (v_1, \dots, v_m) \in \mathcal{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathcal{U}$ be the concatenation of u and v. We endow $\mathcal{U}$ with the lexicographical order, $v \prec w$, if there exists $z \in \mathcal{U}$ such that $v = z(v_1, \dots, v_n)$, $w = z(w_1, \dots, w_m)$ and $v_1 < w_1$.

A plane tree is a non-empty, finite subset $\tau \subset \mathcal{U}$ such that:

(i) $\varnothing \in \tau$;

(ii) if $u \in \tau$ with $|u| \geqslant 1$, then $pr(u) \in \tau$;

(iii) if $u \in \tau$, then there exists an integer $k_u \geqslant 0$ such that $ui \in \tau$ if and only if $1 \leqslant i \leqslant k_u$.

We will view each vertex u of a tree τ as an individual of a population whose τ is the genealogical tree. The vertex $\varnothing$ is called the *root* of the tree, and for every $u \in \tau$, k_u is the number of children (or out-degree) of u (if $k_u = 0$, then u is called a *leaf*), $|u|$ is its *generation*, $pr(u)$ is its *parent* and more generally, the vertices $u, pr(u), pr \circ pr(u), \dots, pr^{|u|}(u) = \varnothing$ are its *ancestors*. Henceforth, unless specified otherwise, *tree* will always be used to mean plane tree.

We define the (planar, but non-rooted) dual tree $T(P)$ of a non-crossing partition P of $[n]$ as follows: we place a black vertex inside each block of the partition and a white vertex inside each other face, then we join two vertices if the corresponding faces share a common edge; here we shall view the singletons as self-loops and the blocks of size two with one double edge. See Figure 2 for an illustration. Observe that the graph thus obtained is indeed a planar tree (meaning that there is an order among all edges adjacent to the same vertex, up to cyclic permutations), with $n+1$ vertices, and that the latter is bipartite: each edge connects two vertices of different colours.

In order to fully recover the partition from the tree (and therefore obtain a bijection), we need to assign a root by distinguishing a *corner* of $T(P)$ (a corner of a vertex in a planar tree is a sector around this vertex delimited by two consecutive edges), thus making it a plane tree. We will do so in two different ways, which will give rise to two different bijections. First, $T^\circ(P)$ is the tree $T(P)$ rooted at the corner of the white vertex that lies in the face containing the vertices 1 and n, and that has the black vertex in the block containing 1 as its first child; $T^\bullet(P)$ is the tree $T(P)$ rooted at the corner of the black vertex in the block containing n and which has the white vertex that lies in the face containing the vertices 1 and n as its first child. See Figures 4 and 5 for an example.

The trees $T^\circ(P)$ and $T^\bullet(P)$ are two-type plane trees: vertices at even generation are coloured in one colour and vertices at odd generation are coloured in another colour. We apply to each a bijection due to Janson and Stefánsson [26, Section 3] which maps such a tree into a one-type tree, that we now describe. This bijection enjoys useful probabilistic features; see Corollary 2.5 below.

We let T denote a plane tree and let $\mathcal{G}(T)$ be its image by this bijection; T and $\mathcal{G}(T)$ have the same vertices but the edges are different. If $T = \{\varnothing\}$ is a singleton, then set

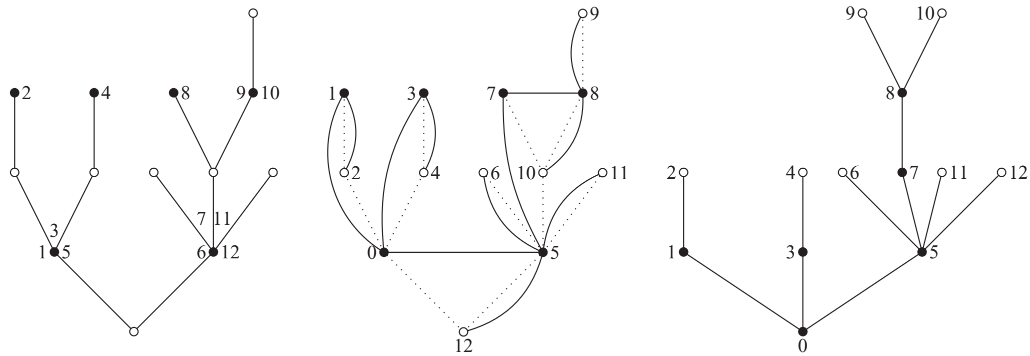


Figure 4. The tree T° associated with the partition from Figure 2, with its black corners indexed according to the contour sequence, and its image $\mathcal{T}^\circ$ by the Janson–Stefánsson bijection, with its vertices indexed in lexicographical order.

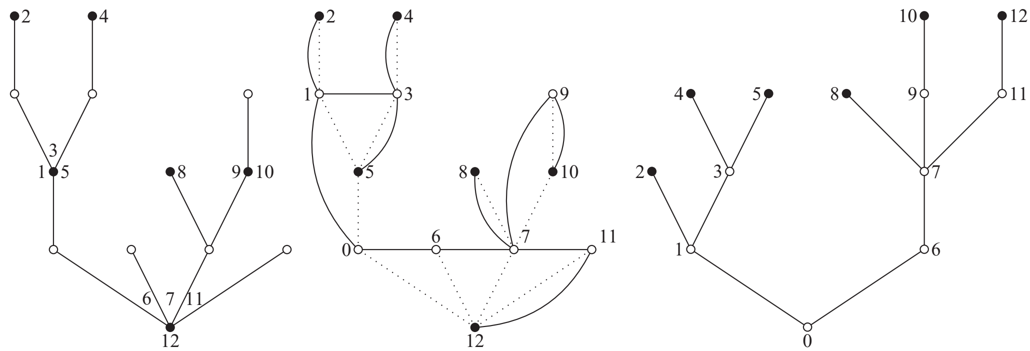


Figure 5. The tree $T^\bullet$ associated with the partition from Figure 2, with its black corners indexed according to the contour sequence, and its image $\mathcal{T}^\bullet$ by the Janson–Stefánsson bijection, with its vertices indexed in lexicographical order.

$\mathcal{G}(T) = \{\varnothing\}$. Otherwise, for every vertex $u \in T$ at even generation with $k_u \geqslant 1$ children, do the following: first, if $u \neq \varnothing$, draw an edge between its parent $pr(u)$ and its first child $u1$, then draw edges between its consecutive children $u1$ and $u2$, $u2$ and $u3, \ldots, u(k_u - 1)$ and uk_u, and finally draw an edge between uk_u and u. If u is a leaf of T, then this procedure reduces to drawing an edge between u and $pr(u)$. We root $\mathcal{G}(T)$ at the first child of the root of T. One can check that $\mathcal{G}(T)$ thus defined is indeed a plane tree, and that the mapping is invertible. Also observe that every vertex at even generation in T is mapped to a leaf of $\mathcal{G}(T)$, and every vertex at odd generation with $k \geqslant 0$ children in T is mapped to a vertex with $k+1$ children in $\mathcal{G}(T)$.

We let

$$\mathcal{T}^\circ(P) := \mathcal{G}(T^\circ(P)) \quad \text{and} \quad \mathcal{T}^\bullet(P) := \mathcal{G}(T^\bullet(P))$$

be the (one-type) trees associated with $T^\circ(P)$ and $T^\bullet(P)$ respectively, and now explain how to reconstruct the non-crossing partition P from the trees $\mathcal{T}^\circ(P)$ and $\mathcal{T}^\bullet(P)$.

To this end, we introduce the notion of a *twig*. If T is a tree and $u,v \in T$, let $[\![u,v]\!]$ denote the shortest path between u and v in T. A twig of T is a set of the form $[\![u,v]\!]$, where u is an ancestor of v and such that all the vertices of $]\!]u,v]\!]$ are the last child of their parent; we agree that $[\![u,u]\!]$ is a twig for every vertex u. Now, if $\tau \in \mathbb{T}_{n+1}$ is a tree, let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ be its vertices listed in lexicographical order. We define two partitions $P_\circ(\tau)$ and $P_\bullet(\tau)$ of $[n]$ as follows:

- $i,j \in [n]$ belong to the same block of $P_\circ(\tau)$ when $u(i)$ and $u(j)$ have the same parent in τ;
- $i,j \in [n]$ belong to the same block of $P_\bullet(\tau)$ when $u(i)$ and $u(j)$ belong to the same twig.

It is an easy exercise to check that for every $\tau \in \mathbb{T}$, $P_\circ(\tau)$ and $P_\bullet(\tau)$ are indeed partitions which, further, are non-crossing. As illustrated by Figures 4 and 5, we have the following result.

Proposition 2.1. *For every non-crossing partition P we have*

$$P = P_\circ(\mathcal{T}^\circ(P)) = P_\bullet(\mathcal{T}^\bullet(P)).$$

Proof. Fix a non-crossing partition P of $[n]$. Let us prove the first equality. Define the *contour sequence* $(u_0,u_1,\ldots,u_{2n})$ of the tree $T^\circ(P)$ as follows: $u_0 = \varnothing$ and for each $i \in \{0,\ldots,2n-1\}$, u_{i+1} is either the first child of u_i which does not appear in the sequence $(u_0,\ldots,u_i)$, or the parent of u_i if all its children already appear in this sequence. Recall that a corner of a vertex $v \in T^\circ(P)$ is a sector around v delimited by two consecutive edges. We index from 1 to n the corners of the black vertices of $T^\circ(P)$, following the contour sequence. By construction of $T^\circ(P)$, we recover P from these corners: for each black vertex of $T^\circ(P)$, the indices of its corners, listed in clockwise order, form a block of P. Now assign labels to the vertices of $\mathcal{T}^\circ(P)$ as follows. By definition of the bijection $\mathcal{G}$, each edge of $\mathcal{T}^\circ(P)$ starts from one of these corners; we then label its other extremity by the label of the corner. The root of $\mathcal{T}^\circ(P)$ is not labelled; we assign it the label 0. The labels thus obtained correspond to the lexicographical order in $\mathcal{T}^\circ(P)$ and the first identity follows.

For the second equality, define the contour sequence of $T^\bullet(P)$ similarly, but starting from the first child of the root, and label the black corners as before. We then label the vertices of $T^\bullet(P)$ as follows: the label of every black vertex is the largest label of its adjacent corners, and then assign the remaining labels of its adjacent corners in decreasing order to its children, starting from the last one. Observe that the root of $T^\bullet(P)$ has as many children as corners, and all the other black vertices have one child less than the number of corners. Thus all the vertices of $T^\bullet(P)$ have labels, except the first child of the root which we label 0. We recover P from $T^\bullet(P)$ as follows: for each black vertex of $T^\bullet(P)$, its label together with the labels of its children form a block of P (and one does not take into account the label 0). As the vertex set of $T^\bullet(P)$ and of $\mathcal{T}^\bullet(P)$ is the same, we also get a labelling of the vertices of $\mathcal{T}^\bullet(P)$. Again, by definition of the $\mathcal{G}$, these labels correspond to the lexicographical order in $\mathcal{T}^\bullet(P)$ and the second identity follows. □

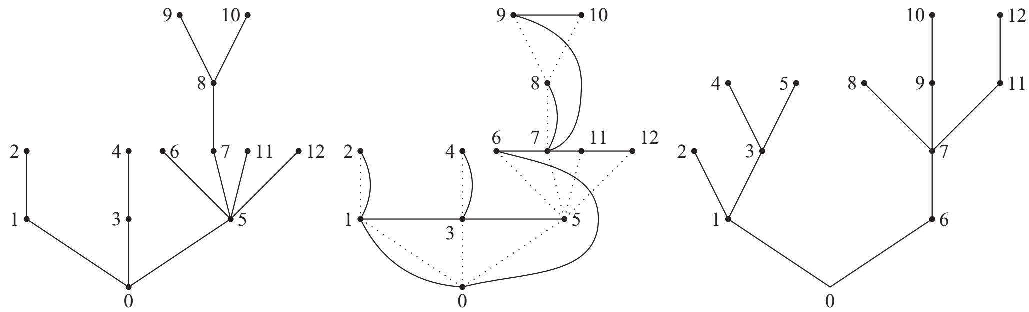


Figure 6. The transformation $\tau \mapsto \mathcal{B}(\tau)$.

Observe from the previous results that the plane trees $\mathcal{T}^\circ(P)$ and $\mathcal{T}^\bullet(P)$ are in bijection. Let us describe a direct operation on trees which maps $\mathcal{T}^\circ(P)$ onto $\mathcal{T}^\bullet(P)$ as depicted in Fig. 6. Starting from a tree $\tau \in \mathbb{T}$, we construct a tree $\mathcal{B}(\tau)$ on the same vertex-set by defining edges as follows: first, we link any two consecutive children in τ; second, we link every vertex v which is the first child of its parent to its youngest ancestor u such that $[\![u, pr(v)]\!]$ is a twig in τ (in this case observe that either u is the root of τ, or v is not the last child of u in $\mathcal{B}(\tau)$). We leave it as an exercise to check that this mapping preserves the lexicographical order.

Proposition 2.2. *For every non-crossing partition P we have*

$$\mathcal{B}(\mathcal{T}^\circ(P)) = \mathcal{T}^\bullet(P).$$

Proof. Fix a non-crossing partition P. Thanks to Proposition 2.1, it is equivalent to showing that

$$P_\bullet(\mathcal{B}(\mathcal{T}^\circ(P))) = P,$$

and we set $P' = P_\bullet(\mathcal{B}(\mathcal{T}^\circ(P)))$ to simplify notation.

Suppose first that $i, j \geqslant 1$ lie in the same block of P. We shall show that i and j belong to the same block of P'. The two corresponding vertices, say, $u(i)$ and $u(j)$, have the same parent in $\mathcal{T}^\circ(P)$. Without loss of generality, assume that $u(i) \prec u(j)$ are consecutive children in $\mathcal{T}^\circ(P)$. It suffices to check that, in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ is the last child of $u(i)$. This simply follows from the fact that $\mathcal{B}$ preserves the lexicographical order and that the children of $u(i)$ in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ excluded, are descendants of $u(i)$ in $\mathcal{T}^\circ(P)$.

Conversely, suppose that $i, j \geqslant 1$ lie in the same block of P'. Without loss of generality, we may assume that, in $\mathcal{B}(\mathcal{T}^\circ(P))$, $u(j)$ is the last child of $u(i)$. We argue by contradiction and assume that, in $\mathcal{T}^\circ(P)$, $u(i)$ and $u(j)$ are not siblings. We saw that in this case, by definition of $\mathcal{B}$, either $u(i)$ is the root, or $u(j)$ is not the last child of $u(i)$ in $\mathcal{B}(\mathcal{T}^\circ(P))$. Both of these cases are excluded. Therefore i and j belong to the same block of P. $\square$

We already mentioned in the Introduction that the bijection $\tau \leftrightarrow P_\circ(\tau)$ was defined by Dershowitz and Zaks [14]; the bijection $\tau \leftrightarrow P_\bullet(\tau)$ was defined by Prodinger [36] and

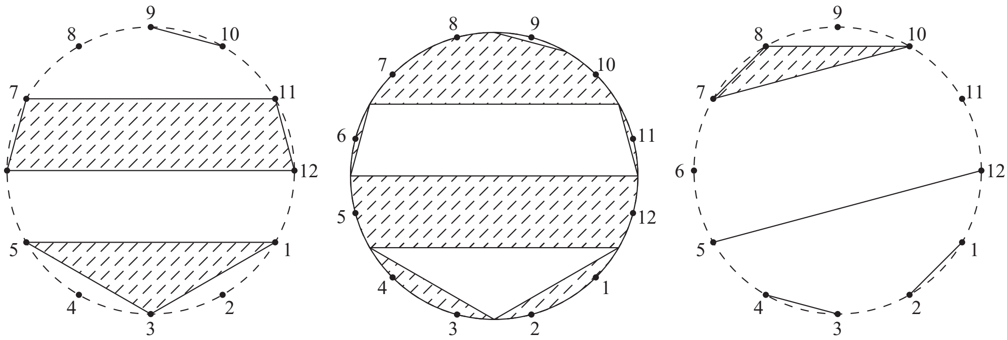


Figure 7. The Kreweras complement of the partition $\{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ is $\{\{1,2\},\{3,4\},\{5,12\},\{6\},\{7,8,10\},\{9\},\{11\}\}$.

further used in combinatorics (see *e.g.* Yano and Yoshida [40]) and in (free) probability (see Ortmann [34]). Roughly speaking, here we unify these two bijections by seeing that (up to the Janson–Stefánsson bijection) they amount to choosing different distinguished corners in the dual two-type planar tree. In this spirit, if P is a non-crossing partition, let us also mention that its *Kreweras complement* $K(P)$ is simply obtained by re-rooting $T(P)$ at a new corner; more precisely, the mappings $(T^\bullet)^{-1} \circ T^\circ$ and $(\mathcal{T}^\bullet)^{-1} \circ \mathcal{T}^\circ$ coincide and both correspond to K.

The Kreweras complement can be formally defined as follows. If we let $\mathbb{NC}(A)$ denote the set of non-crossing partitions on a finite subset $A \subset \mathbb{N}$, then we have canonical isomorphisms

$$\mathbb{NC}_n := \mathbb{NC}(\{1,2,\ldots,n\}) \cong \mathbb{NC}(\{1,3,\ldots,2n-1\}) \cong \mathbb{NC}(\{2,4,\ldots,2n\}).$$

Given two non-crossing partitions $P \in \mathbb{NC}(\{1,3,\ldots,2n-1\})$ and $P' \in \mathbb{NC}(\{2,4,\ldots,2n\})$, one constructs a (possibly crossing) partition $P \cup P'$ of $\{1,2,\ldots,2n\}$. The Kreweras complement of a non-crossing partition $P \in \mathbb{NC}_n \cong \mathbb{NC}(\{1,3,\ldots,2n-1\})$ is then given by

$$K(P) = \max\{P' \in \mathbb{NC}_n \cong \mathbb{NC}(\{2,4,\ldots,2n\}) : P \cup P' \in \mathbb{NC}_{2n}\},$$

where the maximum refers to the partial order of reverse refinement, $P_1 \preceq P_2$, when every block of P_1 is contained in a block of P_2.

The Kreweras complementation can be visualized as follows. Consider the representation of $P \in \mathbb{NC}_n$ in the unit disk as in Figure 2. Invert the colours and rotate the vertices of the regular n-gon by an angle $-\pi/n$. Then the blocks of $K(P)$ are given by the vertices lying in the same 'coloured' component. See Figure 7 for an illustration.

2.2. Simply generated non-crossing partitions and simply generated trees

An important feature of the bijection $\mathcal{B}^\circ : P \mapsto \mathcal{T}^\circ(P)$ is that it transforms simply generated non-crossing partitions into simply generated trees, which were introduced by Meir and Moon [33] and whose definition we now recall.

Given a sequence $w = (w(i); i \geqslant 0)$ of non-negative real numbers, associate with every $\tau \in \mathbb{T}$ a weight $\Omega^w(\tau)$:

$$\Omega^w(\tau) = \prod_{u \in \tau} w(k_u).$$

Then, for every $\tau \in \mathbb{T}_n$, set

$$\mathbb{Q}_n^w(\tau) = \frac{\Omega^w(\tau)}{\sum_{T \in \mathbb{T}_n} \Omega^w(T)}.$$

Again, we always restrict our attention to those values of n for which $\sum_{T \in \mathbb{T}_n} \Omega^w(T) > 0$. A random tree of $\mathbb{T}_n$ sampled according to $\mathbb{Q}_n^w$ is called a *simply generated tree*. A particular case of such trees on which we shall focus in Section 4 is when the sequence of weights w defines a probability measure on $\mathbb{Z}_+$ with mean 1 (see the discussion in Section 3.1 below). In this case, $\mathbb{Q}_n^w$ is the law of a *Galton–Watson tree* with critical offspring distribution w conditioned to have n vertices.

Proposition 2.3. *Let $(w(i); i \geqslant 1)$ be any sequence of non-negative real numbers. Set $w(0) = 1$. Then, for every $P \in \mathbb{NC}_n$,*

$$\mathbb{P}_n^w(P) = \mathbb{Q}_{n+1}^w(\mathcal{T}^\circ(P)).$$

In other words, the bijection $\mathcal{B}^\circ$ transforms simply generated non-crossing partitions into simply generated trees.

Proof. By Proposition 2.1, we have $P = P_\circ(\mathcal{T}^\circ(P))$. In particular, blocks of size $k \geqslant 1$ in P are in bijection with vertices with out-degree k in $\mathcal{T}^\circ(P)$. The claim immediately follows. □

It is also possible to give an explicit description of the law of T° under $\mathbb{P}_n^w$, which turns out to be a two-type simply generated tree. We let $\mathbb{T}^{(\mathsf{e},\mathsf{o})}$ denote the set of finite two-type trees: for every $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$, we let $\mathsf{e}(\tau)$ and $\mathsf{o}(\tau)$, respectively, denote the set of vertices at even and odd generation in τ. Given two sequences of weights w^{e} and w^{o}, we define the weight of tree $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$ by

$$\Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(\tau) = \prod_{u \in \mathsf{e}(\tau)} w^{\mathsf{e}}(k_u) \prod_{u \in \mathsf{o}(\tau)} w^{\mathsf{o}}(k_u).$$

and we define for every $\tau \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$ the set of two-type trees with n vertices,

$$\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\mathsf{o}})}(\tau) = \frac{\Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(\tau)}{\sum_{T \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}} \Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(T)},$$

where, again, we implicitly restrict ourselves to those values of n for which

$$\sum_{T \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}} \Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(T) > 0.$$

A random tree sampled according to $\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\mathsf{o}})}$ is called a *two-type simply generated tree*.

Proposition 2.4. *Let $w = (w(i); i \geqslant 1)$ be a sequence of non-negative real numbers and let $c > 0$ be a positive real number. For every $i \geqslant 0$, set $w^{\mathsf{o}}(i) = w(i+1)$ and $w^{\mathsf{e}}(i) = c^{-(i+1)}$. Then, for every $P \in \mathbb{NC}_n$,*

$$\mathbb{P}_n^w(P) = \mathbb{Q}_{n+1}^{(w^{\mathsf{e}},w^{\mathsf{o}})}(T^\circ(P)).$$

Proof. Fix $P \in \mathbb{NC}_n$; by construction of $T^\circ(P)$ (recall the proof of Proposition 2.1), the vertices at odd generation in $T^\circ(P)$ are in bijection with the blocks of P and the degree of each corresponds to the size of the associated block. Consequently, we have on the one hand

$$\prod_{u\in\mathsf{o}(T^\circ(P))} w^{\mathsf{o}}(k_u) = \prod_{u\in\mathsf{o}(T^\circ(P))} w(k_u+1) = \prod_{B \text{ block of } P} w(\text{size of } B) = \Omega^w(P).$$

On the other hand, since $T^\circ(P) \in \mathbb{T}_{n+1}^{(\mathsf{e},\mathsf{o})}$,

$$\prod_{u\in\mathsf{e}(T^\circ(P))} w^{\mathsf{e}}(k_u) = \prod_{u\in\mathsf{e}(T^\circ(P))} c^{-(k_u+1)} = c^{-\sum_{u\in\mathsf{e}(T^\circ(P))}(k_u+1)} = c^{-(n+1)}.$$

This last term depends only on n and not on P, and the claim follows. □

Recall that $\mathcal{G}$ denotes the Janson–Stefánsson bijection. Then, combining Propositions 2.3 and 2.4, we obtain the following result.

Corollary 2.5. *Let $w = (w(i); i \geqslant 1)$ be any sequence of non-negative real numbers and let $c > 0$ be a positive real number. Set $w(0) = 1$, and for every $i \geqslant 0$ define $w^{\mathsf{o}}(i) = w(i+1)$ and $w^{\mathsf{e}}(i) = c^{-(i+1)}$. Then, for every $T \in \mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$, we have*

$$\mathbb{Q}_n^{w^{\mathsf{e}},w^{\mathsf{o}}}(T) = \mathbb{Q}_n^w(\mathcal{G}(T)).$$

In other words, the Janson–Stefánsson bijection transforms a certain class of two-type simply generated trees into one-type simply generated trees. A similar result implicitly appears in their work [26, Appendix A] in the particular case of Galton–Watson trees, where w^{e} and w^{o} are probability distributions on $\{0,1,\ldots,\}$ and moreover w^{e} is a geometric distribution.

3. Applications

In this section we use simply generated trees to study combinatorial properties of simply generated non-crossing partitions. Indeed, as suggested by Proposition 2.3, it is possible to reformulate questions concerning random non-crossing partitions in terms of random trees, which is more familiar territory.

3.1. Asymptotics of simply generated trees

Following Janson [23], here we describe all the possible regimes arising in the asymptotic behaviour of simply generated trees. All the following discussion appears in [23], but we reproduce it here for the reader's convenience in view of future use, and refer to the latter reference for details and proofs.

Let $(w(i); i \geqslant 0)$ be a sequence of non-negative real numbers with $w(0) > 0$ and $w(k) > 0$ for some $k \geqslant 2$ (and keeping in mind that we will take $w(0) = 1$ in view of Proposition 2.3). Set

$$\Phi(z) = \sum_{k=0}^{\infty} w(k) z^k, \quad \Psi(z) = \frac{z\Phi'(z)}{\Phi(z)} = \frac{\sum_{k=0}^{\infty} k w(k) z^k}{\sum_{k=0}^{\infty} w(k) z^k}, \quad \rho = \left(\limsup_{k \to \infty} w(k)^{1/k} \right)^{-1}.$$

If $\rho = 0$, set $\nu = 0$ and otherwise

$$\nu = \lim_{t \uparrow \rho} \Psi(t).$$

We now define a number $\xi \geqslant 0$ according to the value of ν.

- If $\nu \geqslant 1$, then ξ is the unique number in $(0, \rho]$ such that $\Psi(\xi) = 1$.
- If $\nu < 1$, then we set $\xi = \rho$.

In both cases we have $0 < \Phi(\xi) < \infty$, and we set

$$\pi(k) = \frac{w(k)\xi^k}{\Phi(\xi)} \quad (k \geqslant 0),$$

so π is a probability distribution with expectation $\min(\nu, 1)$ and variance $\xi\Psi'(\xi) \leqslant \infty$. If $\rho = 0$, note that $\pi(0) = 1$.

We say that another sequence of weights $\widetilde{w} = (\widetilde{w}(i); i \geqslant 0)$ is equivalent to w when there exists $a, b > 0$ such that $\widetilde{w}(i) = ab^i w(i)$ for every $i \geqslant 0$. In this case, one can check that $\Omega^{\widetilde{w}}(\tau) = a^n b^{n-1} \Omega^w(\tau)$ for every $\tau \in \mathbb{T}_n$ so that $\mathbb{Q}_n^{\widetilde{w}} = \mathbb{Q}_n^w$ and $\mathbb{P}_n^{\widetilde{w}} = \mathbb{P}_n^w$ for every $n \geqslant 1$. We see that w is equivalent to a probability distribution if and only if $\rho > 0$. In addition, when $\rho > 0$, π defined as above is the unique probability distribution with mean 1 equivalent to w if such a distribution exists; if no such distribution exists, then π is the probability distribution equivalent to w that has the maximal mean.

Example 1. Let $\mathcal{A}$ denote a non-empty subset of $\{1, 2, 3, \ldots\}$ with $\mathcal{A} \neq \{1\}$. Set $w_{\mathcal{A}}(0) = 1$, $w_{\mathcal{A}}(k) = 1$ if $k \in \mathcal{A}$ and $w_{\mathcal{A}}(k) = 0$ if $k \notin \mathcal{A}$. Then the equivalent probability measure $\pi_{\mathcal{A}}$ is defined by

$$\pi_{\mathcal{A}}(k) = \frac{\xi_{\mathcal{A}}^k}{1 + \sum_{i \in \mathcal{A}} \xi_{\mathcal{A}}^i} \mathbb{1}_{k \in \{0\} \cup \mathcal{A}}, \quad \text{where } \xi_{\mathcal{A}} > 0 \text{ is such that } 1 + \sum_{i \in \mathcal{A}} \xi_{\mathcal{A}}^i = \sum_{i \in \mathcal{A}} i \cdot \xi_{\mathcal{A}}^i.$$

For example, for fixed $n \geqslant 1$, we have

$$\pi_{n\mathbb{N}}(k) = \frac{n}{(1+n)^{1+k/n}} \mathbb{1}_{k \in n\mathbb{Z}_+} \quad (k \geqslant 0).$$

In particular, $\pi_{\mathbb{N}}(k) = 1/2^{k+1}$ for every $k \geqslant 0$. Also,

$$\pi_{(2\mathbb{Z}_+ + 1)}(k) = \frac{1 - z^2}{1 + z - z^2} \cdot z^k \cdot \mathbb{1}_{k = 0 \text{ or } k \text{ odd}} \quad (k \geqslant 0),$$

where z is the unique solution of $1 - 2z^2 - 2z^3 + z^4 = 0$ in $[0, 1]$.

Theorem 3.1 (Janson [23], Theorems 7.10 and 7.11). *Let T_n be a random element of $\mathbb{T}_n$ sampled according to $\mathbb{Q}_n^w$, and fix $k \geqslant 0$.*

(i) *We have $\mathbb{P}(k_\varnothing(T_n) = k) \to k\pi(k)$ as $n \to \infty$.*

(ii) *Let $N_k(T_n)$ be the number of vertices with out-degree k in T_n. Then $N_k(T_n)/n$ converges in probability to π_k as $n \to \infty$.*

Note that the quantities $k\pi(k)$ appearing in (i) do not sum to 1 when $v < 1$. It indicates that some mass 'escapes to infinity', so that roughly speaking there is a non-zero probability of the root of T_n having an 'infinite' degree as $n \to \infty$.

Theorem 3.2 (Janson [23], Theorem 18.6). *If $\rho > 0$, we have*

$$\frac{1}{n} \cdot \log\left(\sum_{T \in \mathbb{T}_n} \Omega^w(T) \right) \quad \underset{n\to\infty}{\longrightarrow} \quad \log(\Phi(\xi)/\xi).$$

3.2. Applications in the enumeration of non-crossing partitions with prescribed block sizes
By Proposition 2.1, counting non-crossing partitions of $[n]$ with conditions on the number of blocks of given sizes reduces to counting plane trees of $\mathbb{T}_{n+1}$ with conditions on the number of vertices with given out-degrees, which is a well-trodden path (see *e.g.* [39, Section 5.3]). Since our main interest lies in probabilistic aspects of non-crossing partitions, we shall only give one example of such an application. Let $\mathcal{A}$ be a non-empty subset of $\{1, 2, 3, \ldots\}$ with $\mathcal{A} \neq \{1\}$, and let $\mathbb{NC}_n^{\mathcal{A}}$ denote the set of all non-crossing partitions of $[n]$ with blocks of size only belonging to $\mathcal{A}$. Recall the definition of $\xi_{\mathcal{A}}$ from Example 1.

Proposition 3.3. *Set $\Phi(z) = 1 + \sum_{k \in \mathcal{A}} z^k$. Then*

$$\#\mathbb{NC}_n^{\mathcal{A}} \quad \underset{n\to\infty}{\sim} \quad \gcd(\mathcal{A}) \cdot \sqrt{\frac{\Phi(\xi_{\mathcal{A}})}{2\pi\Phi''(\xi_{\mathcal{A}})}} \cdot \left(\frac{\Phi(\xi_{\mathcal{A}})}{\xi_{\mathcal{A}}} \right)^{n+1} \cdot n^{-3/2},$$

where $n \to \infty$ in such a way that n is divisible by $\gcd(\mathcal{A})$.

Proof. Setting $\overline{\mathcal{A}} = \{0\} \cup \mathcal{A}$, observe that $\#\mathbb{NC}_n^{\mathcal{A}} = \#\mathbb{T}_{n+1}^{\overline{\mathcal{A}}}$ by Proposition 2.1. But, by [20, Proposition I.5.], the generating function

$$T^{\overline{\mathcal{A}}}(z) = \sum_{n \geqslant 1} \#\mathbb{T}_n^{\overline{\mathcal{A}}} \cdot z^n$$

satisfies the implicit equation $T^{\overline{\mathcal{A}}}(z) = z\Phi(T^{\overline{\mathcal{A}}}(z))$. The claim then immediately follows from [20, Theorem VII.2 and Remark VI.17]. □

Let us mention that explicit expressions for $\#\mathbb{NC}_n^{\mathcal{A}}$ for n fixed are known for two particular choices of $\mathcal{A}$. Edelman [19] has found an explicit formula for $\#\mathbb{NC}_{kn}^{k\mathbb{Z}_+}$ (*i.e.* for k-divisible non-crossing partitions) and Arizmendi and Vargas [4] have found the explicit expression of $\#\mathbb{NC}_{kn}^{\{k\}}$ (*i.e.* for k-equal non-crossing partitions):

$$\#\mathbb{NC}_{kn}^{\{k\}} = \frac{1}{(k-1)n+1} \binom{kn}{n} \quad \text{and} \quad \#\mathbb{NC}_{kn}^{k\mathbb{Z}_+} = \frac{1}{kn+1} \binom{(k+1)n}{n}.$$

3.3. Applications in free probability

Recall from the Introduction the definition of the R-transform R_μ of a compactly supported probability measure μ on the real line, and that it is related to its associated free cumulants $(\kappa_i(\mu); i \geqslant 1)$ by the formula

$$R_\mu(z) = \sum_{n=0}^{\infty} \kappa_{n+1}(\mu) z^n.$$

Theorem 3.4. *Let μ be a compactly supported probability measure on $\mathbb{R}$ different from a Dirac mass. Assume that its free cumulants $(\kappa_i(\mu); i \geqslant 1)$ are all non-negative. Set*

$$\rho = \left(\limsup_{n\to\infty} \kappa_n(\mu)^{1/n}\right)^{-1} \quad \textit{and} \quad \nu = 1 + \lim_{t\uparrow\rho} \frac{t^2 R'_\mu(t) - 1}{tR_\mu(t)+1}.$$

(i) *If $\nu \geqslant 1$, there exists a unique number ξ in $(0,\rho]$ such that $R'_\mu(\xi) = 1/\xi^2$ and*

$$\frac{1}{n} \cdot \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t) \quad \underset{n\to\infty}{\longrightarrow} \quad \log\left(\frac{1}{\xi} + R_\mu(\xi)\right).$$

(ii) *If $\nu < 1$, we have*

$$\frac{1}{n} \cdot \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t) \quad \underset{n\to\infty}{\longrightarrow} \quad \log\left(\frac{1}{\rho} + R_\mu(\rho)\right).$$

Note that the equality $R'_\mu(\xi) = 1/\xi^2$ is equivalent to $K'_\mu(\xi) = 0$, where we recall that K_μ denotes the inverse of the Cauchy transform of μ.

Proof. First note that $\rho > 0$, as R_μ is analytic on a neighbourhood of the origin. We then apply the results of Section 3.1 with weights w defined by $w(0) = 1$ and $w(i) = \kappa_i(\mu)$ for $i \geqslant 1$. The fact that μ is different from a Dirac mass guarantees that $w(k) > 0$ for some $k \geqslant 2$. Observe that

$$\Phi(z) = 1 + zR_\mu(z) = zK_\mu(z) \quad \text{and} \quad \Psi(z) = 1 + \frac{z^2 R'(z) - 1}{zR(z)+1}.$$

In particular, $\Psi(z) = 1$ if and only if $R'_\mu(z) = 1/z^2$. The claim then follows by combining (1.1) with Theorem 3.2. $\square$

See Example 2 below for an example where $\nu < 1$. If μ is the uniform measure on $[0,1]$, its free cumulants are not all non-negative, as $R_\mu(z) = 1/(1-\mathrm{e}^{-z}) - 1/z$. See also [5] for information concerning Taylor series of the R-transform of measures which are not compactly supported.

Proof of Theorem 1.2. Let s_μ be the maximum of the support of a compactly supported probability measure μ on $\mathbb{R}$. It is well known and simple to check that

$$\log(s_\mu) = \limsup_{n\to\infty} \frac{1}{n} \log \int_{\mathbb{R}} t^n \mu(\mathrm{d}t).$$

Hence, taking into account (1.1), the claim immediately follows from Theorem 3.4. □

Example 2. We apply Theorem 1.2 to four different measures μ.

(i) If $\mu(\mathrm{d}x) = 1/(\pi\sqrt{1-x^2})\mathbb{1}_{|x|\leqslant 1}\mathrm{d}x$ is the arcsine law (which is also the free additive convolution $\lambda \boxplus \lambda$ with $\lambda = (\delta_{-1/2} + \delta_{1/2})/2)$), we have $\rho = \infty$, $v = 0$, so that

$$R_\mu(z) = \left(\sqrt{1+z^2} - 1\right)/z,$$

and we recover that $s_\mu = 1/\infty + R(\infty) = 1$.

(ii) If μ is the free convolution of a free Poisson law of parameter 1 and the uniform distribution on $[-1, 1]$, then $R_\mu(z) = \coth(z) - z^{-1} + (1-z)^{-1}$, $\rho = 1$, $v = \infty$ so that

$$s_\mu = \coth(z_*) + \frac{1}{1-z_*} \simeq 4.16, \quad \text{where } \operatorname{csch}(z_*)(1-z_*)^2 = 1 \text{ with } z_* \in (0,1).$$

This gives a simpler expression than that of [34, Example 6.2], which involves solutions of two implicit equations.

(iii) If μ is such that $R_\mu(z) = 1/z - \pi\coth(\pi z)$ (this corresponds to the Lévy area corresponding to the free Brownian bridge introduced in [35]), then

$$s_\mu = \frac{2 - \sqrt{2 - \pi^2 z_*^2}}{z_*} \simeq 3.94, \quad \text{where } \frac{\sin(\pi z_*)}{\pi z_*} = \frac{\sqrt{2}}{2} \text{ with } z_* \in (0,1).$$

This gives a simpler expression than that of [34, Proposition 5.12],

(iv) As noted by Ortmann [34, Section 6.1], if λ is a finite compactly supported measure on $\mathbb{R}$ and $\alpha \in \mathbb{R}$, by [6] or [22, Theorem 3.3.6], there exists a compactly supported probability measure μ such that

$$R_\mu(z) = \alpha + \int \frac{z}{1-xz}\lambda(\mathrm{d}x),$$

and all the cumulants of μ are non-negative, so that Theorem 3.4 and Theorem 1.2 apply to the corresponding normalized probability measure. This in fact corresponds to the class of so-called freely infinitely divisible measures.

In particular, if $\lambda(\mathrm{d}x) = c(1-x)^\alpha \mathbb{1}_{0\leqslant x\leqslant 1}\mathrm{d}x$ with $c > 0, \alpha > 1$, then μ is such that

$$R_\mu(z) = \int_{\mathbb{R}} \frac{z}{1-xz}\lambda(\mathrm{d}x)$$

and

$$\kappa_n(\mu) = c\frac{\Gamma(1+\alpha)\cdot\Gamma(n-1)}{\Gamma(n+\alpha)}\mathbb{1}_{n\geqslant 2}, \quad \rho = 1, \quad v = \frac{(2\alpha-1)c}{(\alpha-1)(\alpha+c)}.$$

Note that $\kappa_n(\mu) \sim c\Gamma(1+\alpha)\cdot n^{-1-\alpha}$ as $n \to \infty$ and that $v = 1$ if and only if $c = \alpha - 1$. For example, for $\alpha = 2$ and $c = 1/2$, we have $v = 3/5 < 1$ and $s_\mu = 1 + R_\mu(1) = 5/4$.

3.4. Distribution of the block sizes in random non-crossing partitions

We are now interested in the distribution of block sizes in large simply generated non-crossing partitions. We fix a sequence of non-negative weights $w = (w(i); i \geqslant 1)$ such that $w(k) > 0$ for some $k \geqslant 2$. Set $w(0) = 1$, and let P_n be a random non-crossing partition with law $\mathbb{P}_n^w$. Let π denote the probability distribution which corresponds to the weights w in the sense of Section 3.1. Finally, set $T_{n+1} = \mathcal{T}^\circ(P_n)$, so that by Proposition 2.3, T_{n+1} is a simply generated tree with $n+1$ vertices with law $\mathbb{Q}_{n+1}^w$.

3.4.1. Blocks of given size. If P is a non-crossing partition and A is a non-empty subset of $\mathbb{N}$, we let $\zeta_A(P)$ be the number of blocks of P whose size belongs to A. In particular, notice that $\zeta_{\mathbb{N}}(P)$ is the total number of blocks of P.

Theorem 3.5.

(i) *Let $S_1(P_n)$ be the size of the block containing 1 in P_n. Then, for every $k \geqslant 1$,*

$$\mathbb{P}(S_1(P_n) = k) \to k\pi(k) \quad \text{as } n \to \infty.$$

(ii) *Let B_n be a block chosen uniformly at random in P_n. Assume that $\pi(0) < 1$. Then, for every $k \geqslant 1$,*

$$\mathbb{P}(|B_n| = k) \to \pi(k)/(1 - \pi(0)) \quad \text{as } n \to \infty.$$

(iii) *Let A be a non-empty subset of $\mathbb{N}$. As $n \to \infty$, the convergence $\zeta_A(P_n)/n \to \pi(A)$ holds in probability and, in addition,*

$$\mathbb{E}[\zeta_A(P_n)]/n \to \pi(A).$$

In particular, the total number of blocks of P_n is of order $(1 - \pi(0))n$ when $\pi(0) < 1$. Note again that in the case $\nu < 1$, the quantities $k\pi(k)$ appearing in (i) do not sum to 1, so that roughly speaking there is a non-zero probability of the size of the block containing 1 in P_n being 'infinite' as $n \to \infty$.

Proof. For (i), simply note that $S_1(P_n) = k_\varnothing(T_{n+1})$, and the claim immediately follows from Theorem 3.1(i). For the second assertion, if T is a tree, let $N_k(T)$ denote the number of vertices of T with out-degree k. Note that B_n has the law of the out-degree of an internal (*i.e.* not a leaf) vertex of T_{n+1} chosen uniformly at random. As a consequence,

$$\mathbb{P}(|B_n| = k) = \mathbb{E}\left[\frac{N_k(T_{n+1})}{n - N_0(T_{n+1})}\right].$$

By Theorem 3.1(ii), $N_k(T_{n+1})/(n - N_0(T_{n+1}))$ converges in probability to $\pi(k)/(1 - \pi(0))$ as $k \to \infty$, and is clearly bounded by 1. The second assertion then follows from the dominated convergence theorem. For the last assertion, observe that $\zeta_A(P_n) = N_A(T_{n+1})$, where $N_A(T_{n+1})$ denotes the number of vertices of T_{n+1} with out-degree in A. Then, fix $K \geqslant 1$, and to simplify notation, set $A_K = A \cap [K]$, so that by Theorem 3.1(ii), the convergence $\zeta_{A_K}(P_n)/n \to \pi(A_K)$ holds in probability as $n \to \infty$. Since $|\zeta_A(P_n) - \zeta_{A_K}(P_n)| \leqslant n/K$, the quantity $|\zeta_A(P_n)/n - \zeta_{A_K}(P_n)/n|$ can be made arbitrarily small by choosing K sufficiently large. It follows that $\zeta_A(P_n)/n \to \pi(A)$ in probability as $n \to \infty$, and the last claim readily follows by the dominated convergence theorem. □

In the case $\pi(0) = 1$ (which corresponds to $\rho = 0$), (i) tells us that the convergence $S_1(P_n) \to \infty$ holds in probability as $n \to \infty$, but the asymptotic behaviour of $|B_n|$ and the total number of blocks of P_n remains unclear. Unfortunately, it seems that one cannot say anything more in full generality. Indeed:

(i) If $w(k) = k!^{\alpha}$ with $\alpha > 1$, by [25, Remark 2.9], with probability tending to one as $n \to \infty$, the root of T_{n+1} has n children which are all leaves. Therefore, as $n \to \infty$, $\mathbb{P}(S_1(P_n) = n) \to 1$, $\mathbb{P}(|B_n| = n) \to 1$ and $\mathbb{P}(\zeta_{\mathbb{N}}(P_n) = 1) \to 1$.

(ii) If $w(k) = k!$, by [25, Theorem 2.4], with probability tending to one as $n \to \infty$, the root of T_{n+1} has $n - U_{n+1}$ children which are all leaves, except U_{n+1} of them (which have only one vertex grafted onto them), and U_{n+1} converges in distribution to X, a Poisson random variable of parameter 1, as $n \to \infty$. Therefore, as $n \to \infty$, $n - S_1(P_n) \to X$ in distribution,

$$\mathbb{P}(|B_n| = 1) \to \mathbb{E}[X/(X+1)] = 1/e, \quad \mathbb{P}(|B_n| = S_1(P_n)) \to 1 - 1/e \quad \text{and} \quad \zeta_{\mathbb{N}}(P_n) \to X+1$$

in distribution.

(iii) If $w(k) = k!^{\alpha}$ with $0 < \alpha < 1$ and $1/\alpha \notin \mathbb{N}$ for simplicity, by [25, Theorem 2.5], as $n \to \infty$, $k_{\varnothing}(T_{n+1})/n \to 1$ in probability, for every $1 \leqslant i \leqslant \lfloor 1/\alpha \rfloor$, $N_i(T_n)/n^{1-i\alpha} \to i!^{\alpha}$ in probability and, with probability tending to one as $n \to \infty$, $N_i(T_n) = 0$ for every $i > \lfloor 1/\alpha \rfloor$. Therefore, as $n \to \infty$, $S_1(P_n)/n \to 1$ in probability. Also, noting that

$$\mathbb{P}(|B_n| = k) = \mathbb{E}\left[\frac{N_k(T_{n+1})}{\sum_{i \geqslant 1} N_i(T_{n+1})}\right], \quad \zeta_{\mathbb{N}}(P_n) = \sum_{i \geqslant 1} N_i(T_{n+1}),$$

we get that $\mathbb{P}(|B_n| = 1) \to 1$ and $\zeta_{\mathbb{N}}(P_n)/n^{1-\alpha} \to 1$ in probability.

In addition, [23, Example 19.39] gives an example where $\rho = 0$ and $k_{\varnothing}(T_n)/n \to 0$ in probability.

3.4.2. Asymptotic normality of the block sizes. Theorem 3.5(ii) shows that a law of large numbers holds for $\zeta_A(P_n)$. Under some additional regularity assumptions on the weights, it is possible to obtain a central limit theorem. Specifically, assume that w is equivalent (in the sense of Section 3.1) to a probability distribution π which is critical (meaning that its mean is equal to 1) and has finite positive variance σ^2. In this case, the following result holds.

Theorem 3.6. *Fix an integer $k \geqslant 1$, and let $A_1, \ldots, A_k$ be non-empty subsets of $\mathbb{N}$. Then there exists a centred Gaussian vector $(X_{A_1}, \ldots, X_{A_k})$ such that the convergence*

$$\left(\frac{\zeta_{A_1}(P_n) - \pi(A_1)n}{\sqrt{n}}, \ldots, \frac{\zeta_{A_k}(P_n) - \pi(A_k)n}{\sqrt{n}}\right) \quad \xrightarrow[n\to\infty]{(d)} \quad (X_{A_1}, \ldots, X_{A_k})$$

holds in distribution. In addition, we have

$$\mathbb{E}[X_{A_i}^2] = \pi(A_i)(1 - \pi(A_i)) - \frac{1}{\sigma^2} \sum_{r \in A_i} (r-1)^2 \pi(r) \quad \textit{for } 1 \leqslant i \leqslant k$$

and

$$\mathrm{Cov}(X_{A_i}, X_{A_j}) = -\pi(A_i)\pi(A_j) - \frac{1}{\sigma^2} \sum_{r \in A_i} (r-1)^2 \pi(r) \cdot \sum_{s \in A_j} (s-1)^2 \pi(s)$$

if $1 \leqslant i \neq j \leqslant k$ are such that $A_i \cap A_j = \emptyset$.

This result is just a translation of the corresponding known result for conditioned Galton–Watson trees. Recalling that $T_{n+1} = \mathcal{T}^\circ(P_n)$, let $N_A(T_{n+1})$ denote the number of vertices of T_{n+1} with out-degree in A. Then $(\zeta_{A_1}(P_n), \ldots, \zeta_{A_k}(P_n))$ is equal to $(N_{A_1}(T_{n+1}), \ldots, N_{A_k}(T_{n+1}))$, and Theorem 3.6 then follows from [24, Example 2.2] (in this reference, the results are stated when $\#A_i = 1$ for every i, but it is a simple matter to see that they still hold).

3.4.3. Large deviations for the empirical block size distribution. Let $\mathcal{M}_n$ denote the law of the size of a block of P_n, chosen uniformly at random among all possible blocks, so that $\mathcal{M}_n$ is a random probability measure on $\mathbb{N}$. Dembo, Mörters and Sheffield [13, Theorem 2.2] establish a large deviation principle for the empirical out-degree distribution in Galton–Watson trees. Therefore, we believe that an analogous large deviation principle holds for $\mathcal{M}_n$ (at least when the weights are equivalent to a critical probability distribution having a finite exponential moment), which would in particular extend a result of Ortmann [34, Theorem 1.1], who established such a large deviation principle in the case of uniformly distributed k-divisible non-crossing partitions. The point is that Ortmann uses the bijection $P \leftrightarrow \mathcal{T}^\bullet(P_n)$, but we believe that it is simpler to use the bijection $P \leftrightarrow \mathcal{T}^\circ(P)$ since $\mathcal{T}^\circ(P_n)$ is a simply generated tree but in general $\mathcal{T}^\bullet(P_n)$ is not. However, we have not worked out the details.

3.4.4. Largest blocks. Depending on the weights, Janson [23, Sections 9 and 19] obtains general results concerning the largest out-degrees of simply generated trees. Since the sequence of out-degrees of vertices of T_{n+1} that are not leaves, listed in non-increasing order, is equal to the sequence of sizes of blocks of P_n, listed in non-increasing order, one gets estimates on the sizes of the largest blocks of P_n. We do not enter into details, but refer to [23] for precise statements.

3.4.5. Local behaviour. Theorem 3.5(i) describes the distributional limit of the size of the block of P_n containing 1; it is also possible to describe the behaviour of the blocks at 'finite distance' of the latter. Indeed, as we have seen in Section 2.2, when P_n is sampled according to $\mathbb{P}_n^w$, then its two-type dual tree $T_n^\circ = T^\circ(P_n)$ is distributed according to $\mathbb{Q}_{n+1}^{(w^{\mathsf{e}},w^{\mathsf{o}})}$, where $w^{\mathsf{o}}(i) = w(i+1)$ and $w^{\mathsf{e}}(i) = 1$ for every $i \geqslant 0$. In this case, for every tree $\tau \in \mathbb{T}^{(\mathsf{e},\mathsf{o})}$ we have

$$\Omega^{(w^{\mathsf{e}},w^{\mathsf{o}})}(\tau) = \prod_{u \in \mathsf{e}(\tau)} w^{\mathsf{e}}(k_u) \prod_{u \in \mathsf{o}(\tau)} w^{\mathsf{o}}(k_u) = \prod_{u \in \mathsf{o}(\tau)} w(\deg(u)),$$

and Björnberg and Stefánsson [10, Theorem 3.1] have obtained a limit theorem for the measure $\mathbb{Q}_n^{(w^{\mathsf{e}},w^{\mathsf{o}})}$ on $\mathbb{T}_n^{(\mathsf{e},\mathsf{o})}$ as $n \to \infty$, in the local topology. Loosely speaking, the dual tree T_n° converges locally to a limiting infinite two-type tree which can be explicitly constructed, and which is in a certain sense a two-type Galton–Watson tree conditioned to survive. We do not enter into details as we will not use this; we refer to [10] for precise statements and proofs.

4. Non-crossing partitions as compact subsets of the unit disk

We investigate in this section the asymptotic behaviour, as $n \to \infty$, of a non-crossing partition sampled according to $\mathbb{P}_n^\mu$ and viewed as an element of the space of all compact subsets of the unit disk equipped with the Hausdorff distance.

Main assumptions. We restrict ourselves to the case where $\mu = (\mu(k); k \geqslant 0)$ defines a critical probability measure, that is, $\sum_{k=0}^\infty \mu(k) = \sum_{k=0}^\infty k\mu(k) = 1$. Recall from Section 3.1 that any sequence of weights $(w(k); k \geqslant 0)$ such that

$$\rho = \left(\limsup_{k\to\infty} w(k)^{1/k} \right)^{-1} > 0 \quad \text{and} \quad \lim_{t\uparrow\rho} \frac{\sum_{k=0}^\infty k w(k) t^k}{\sum_{k=0}^\infty w(k) t^k} \geqslant 1$$

is equivalent to such a measure μ and then $\mathbb{P}_n^\mu = \mathbb{P}_n^w$ for every $n \geqslant 1$. We shall in addition assume that μ belongs to the domain of attraction of a stable law of index $\alpha \in (1,2]$, that is, either it has finite variance, $\sum_{k=0}^\infty k^2\mu(k) < \infty$ (in the case $\alpha = 2$), or $\sum_{k=j}^\infty \mu(k) = j^{-\alpha}L(j)$, where L is a slowly varying function at infinity. Without further notice, we always assume that $\mu(0) + \mu(1) < 1$ to discard degenerate cases.

In this section we shall establish the following result.

Theorem 4.1. *Fix $\alpha \in (1,2]$. There exists a random compact subset of the unit disk $\mathbf{L}_\alpha$ such that for every critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, if P_n is a random non-crossing partition sampled according to $\mathbb{P}_n^\mu$, for every integer $n \geqslant 1$ such that $\mathbb{P}_n^\mu$ is well defined, the convergence*

$$P_n \xrightarrow[n\to\infty]{(d)} \mathbf{L}_\alpha$$

holds in distribution for the Hausdorff distance on the space of all compact subsets of $\overline{\mathbb{D}}$.

The random compact set $\mathbf{L}_\alpha$ is a geodesic lamination; for $\alpha = 2$, the set $\mathbf{L}_2$ is Aldous's Brownian triangulation of the disk [2], while $\mathbf{L}_\alpha$ is the α-stable lamination introduced in [27] for $\alpha \in (1,2)$. Observe that Theorem 4.1 applies to uniform $\mathcal{A}$-constrained non-crossing partitions of $[n]$ when $\mathcal{A} \neq \{1\}$, since this law is $\mathbb{P}_n^{w_\mathcal{A}}$ where $w_\mathcal{A}(k) = 1$ if $k \in \mathcal{A}$ and $w_\mathcal{A}(k) = 0$ otherwise; the equivalent probability distribution defined in Example 1 is then critical and with finite variance, and thus corresponds to $\alpha = 2$.

Before explaining the construction of $\mathbf{L}_\alpha$, we mention an interesting corollary. Recall from the Introduction the notation $\mathsf{C}(P_n)$ for the (angular) length of the longest chord.

Corollary 4.2. *Fix $\alpha \in (1,2]$. There exists a random variable C_α such that for every critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, if P_n is a random non-crossing partition sampled according to $\mathbb{P}_n^\mu$, for every integer $n \geqslant 1$ such that $\mathbb{P}_n^\mu$ is well defined, the convergence*

$$\mathsf{C}(P_n) \xrightarrow[n\to\infty]{(d)} \mathsf{C}_\alpha$$

holds in distribution.

This immediately follows from Theorem 4.1, since the functional 'longest chord' is continuous on the set of laminations. Aldous [2] (see also [16]) showed that the law of C_2 has the following explicit distribution:

$$\frac{1}{\pi}\frac{3x-1}{x^2(1-x)^2\sqrt{1-2x}}\mathbb{1}_{1/3\leqslant x\leqslant 1/2}\mathrm{d}x.$$

See [37] for a study of the longest chord of stable laminations. As before, observe that Theorem 1.3 follows from Corollary 4.2, which applies with $\alpha = 2$ for uniform $\mathcal{A}$-constrained non-crossing partitions of $[n]$ when $\mathcal{A} \neq \{1\}$.

Techniques. We briefly comment on the main techniques involved in the proof of Theorem 4.1. Since it is simple to recover P_n from its dual two-type tree $T^\circ(P_n)$, it seems natural to study scaling limits of $T^\circ(P_n)$. However, this is not the road we take: we instead code P_n by the associated one-type tree $\mathcal{T}^\circ(P_n)$, which, as we have earlier seen, has the law of a Galton–Watson tree with offspring distribution μ conditioned to have $n+1$ vertices, and is therefore simpler to study. We then follow the route of [27]: we code $\mathcal{T}^\circ(P_n)$ via a discrete walk; the latter converges in distribution to a continuous-time process, we then define $\mathbf{L}_\alpha$ from this limit path and we show that it is indeed the limit of the discrete non-crossing partitions.

In [27], it is shown that certain random dissections of $[n]$ (a dissection of a polygon with n vertices is a collection of non-crossing diagonals) are shown to converge to the stable lamination, by using the fact that their dual trees are Galton–Watson trees conditioned to have a fixed number of leaves. Our arguments are similar to that of [27, Sections 2 and 3], but the devil is in the details since the objects under consideration and their coding by trees are different: first, vertices with out-degree 1 are forbidden in [27], and second a vertex with out-degree k in [27] corresponds to $k+1$ chords in the associated discrete lamination, whereas in our case a vertex with out-degree k corresponds to k chords in the associated non-crossing partition. In particular, the proofs of [27, Sections 2 and 3] do not apply with minor modifications, and for this reason we give a complete proof of Theorem 4.1.

From now on, we fix $\alpha \in (1, 2]$, a critical offspring distribution μ belonging to the domain of attraction of a stable law of index α, and we let P_n be a random non-crossing partition sampled according to $\mathbb{P}_n^\mu$, for every integer $n \geqslant 1$ such that $\mathbb{P}_n^\mu$ is well defined.

4.1. Non-crossing partitions and paths

We first explain how a plane tree can be coded by a function called a *Łukasiewicz path*, and then we describe how to define a non-crossing partition P from the Łukasiewicz path coding the tree $\mathcal{T}^\circ(P)$.

Let $\tau \in \mathbb{T}_{n+1}$ and let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ be its vertices, listed in lexicographical order. Recall that k_u denotes the number of children of $u \in \tau$. The Łukasiewicz path $\mathcal{W}(\tau) = (\mathcal{W}_j(\tau); 0 \leqslant j \leqslant n+1)$ of τ is defined by $\mathcal{W}_0(\tau) = 0$ and

$$\mathcal{W}_{j+1}(\tau) = \mathcal{W}_j(\tau) + k_{u(j)}(\tau) - 1 \quad \text{for every } 0 \leqslant j \leqslant n.$$

One easily checks that $\mathcal{W}_j(\tau) \geqslant 0$ for every $0 \leqslant j \leqslant n$ but $\mathcal{W}_{n+1}(\tau) = -1$. Observe that for every $0 \leqslant j \leqslant n$, $\mathcal{W}_{j+1}(\tau) - \mathcal{W}_j(\tau) \geqslant -1$, with equality if and only if $u(j)$ is a leaf of τ. The next result, whose proof is left as an exercise, explains how to reconstruct a plane tree from its Łukasiewicz path.

Proposition 4.3. *Let $\tau \in \mathbb{T}_{n+1}$, let $\varnothing = u(0) \prec u(1) \prec \cdots \prec u(n)$ be its vertices listed in lexicographical order and let $\mathcal{W}(\tau)$ be its Łukasiewicz path. Fix an integer $0 \leqslant j \leqslant n-1$ such that $k := k_{u(j)}(\tau) \geqslant 1$. Let $s_1, \ldots, s_k \in \{1, \ldots, n\}$ be defined by*

$$s_\ell = \inf\{m \geqslant j+1 : \mathcal{W}_m(\tau) = \mathcal{W}_{j+1}(\tau) - (\ell - 1)\} \quad \textit{for } 1 \leqslant \ell \leqslant k$$

(in particular, $s_1 = j+1$). Then $u(s_1), u(s_2), \ldots, u(s_k)$ are the children of $u(j)$ listed in lexicographical order.

We now describe how to define a non-crossing partition from a Łukasiewicz path. Fix $n \in \mathbb{N}$ and $W = (W_j; 0 \leqslant j \leqslant n+1)$ a path such that $W_0 = 0$, for every $0 \leqslant j \leqslant n$, $W_{j+1} - W_j \geqslant -1$ with the condition that $W_j \geqslant 0$ for every $0 \leqslant j \leqslant n$ and $W_{n+1} = -1$. Define

$$k_j = W_{j+1} - W_j + 1 \quad \text{for every } 0 \leqslant j \leqslant n-1.$$

If $k_j \geqslant 1$, then let

$$s^j_\ell = \inf\{m \geqslant j+1 : W_m = W_{j+1} - (\ell - 1)\} \quad \text{for every } 1 \leqslant \ell \leqslant k_j,$$

and then set $s^j_{k_j+1} = s^j_1 = j+1$. Next define $\mathbf{P}(W)$ by

$$\mathbf{P}(W) = \bigcup_{j:k_j \geqslant 1} \bigcup_{\ell=1}^{k_j} \left[\exp\left(-2i\pi \frac{s^j_\ell}{n}\right), \exp\left(-2i\pi \frac{s^j_{\ell+1}}{n}\right)\right]. \tag{4.1}$$

Let us briefly explain what this means: if W is the Łukasiewicz path of a tree τ with its vertices labelled as above, then k_j is the number of children of $u(j)$, and $s^j_1, \ldots, s^j_{k_j}$ are the indices of its children. Recall from Section 2.1 that from a tree τ, we can define a non-crossing partition $P_\circ(\tau)$ by joining two consecutive children in τ (where the first and last ones are consecutive by convention); this is exactly what is done in (4.1). Recall also from Section 2.1 the construction of the plane tree tree $\mathcal{T}^\circ(P)$ from a non-crossing partition P.

Proposition 4.4. *For every non-crossing partition P, we have*

$$P = \mathbf{P}(\mathcal{W}(\mathcal{T}^\circ(P))).$$

Proof. To simplify notation, let $\mathcal{W}^\circ$ denote the Łukasiewicz path of $\mathcal{T}^\circ(P)$ and let n be its length. First, note that $\mathbf{P}(\mathcal{W}^\circ)$ is a partition of $[n]$: with the notation used in (4.1), the blocks are given by the sets $\{s^j_1, \ldots, s^j_{k_j}\}$ for the j such that $k_j \geqslant 1$. To show that it is non-crossing, fix $j, j' \in \{0, \ldots, n-1\}$ with $k_j, k_{j'} \geqslant 1$ and fix $\ell \in \{1, \ldots, k_j + 1\}$ and $\ell' \in \{1, \ldots, k_{j'} + 1\}$ with $(j, \ell) \neq (j', \ell')$; one checks that the intervals $(s^j_\ell, s^j_{\ell+1})$ and

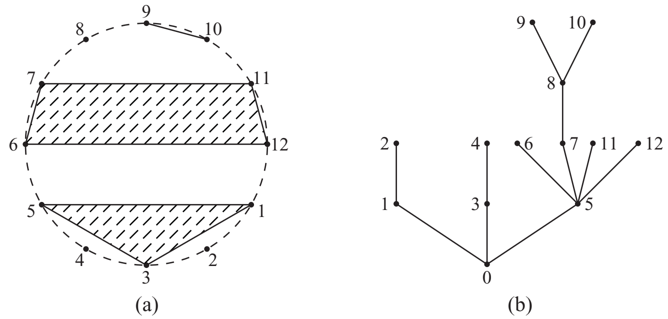


Figure 8. (a) The partition $P = \{\{1,3,5\},\{2\},\{4\},\{6,7,11,12\},\{8\},\{9,10\}\}$ and (b) the tree $\mathcal{T}^{\circ}(P)$.

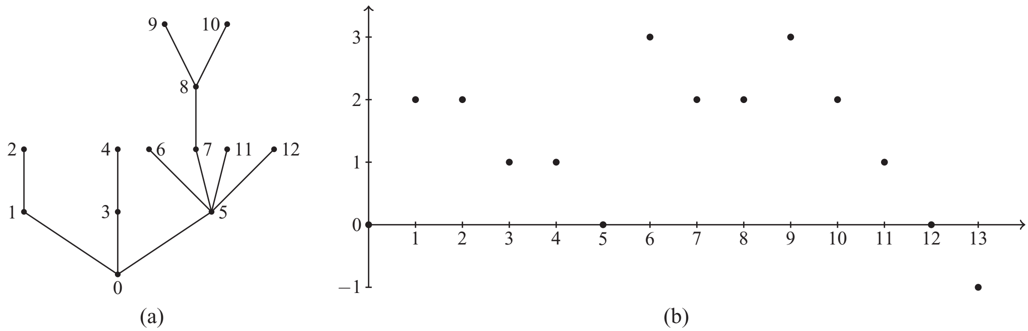


Figure 9. (a) A plane tree and (b) its Łukasiewicz path.

$(s^{j'}_{\ell'}, s^{j'}_{\ell'+1})$ are either disjoint or one is included in the other so that the chords

$$\left[\exp\left(-2\mathrm{i}\pi\frac{s^{j}_{\ell}}{n}\right), \exp\left(-2\mathrm{i}\pi\frac{s^{j}_{\ell+1}}{n}\right)\right] \quad \text{and} \quad \left[\exp\left(-2\mathrm{i}\pi\frac{s^{j'}_{\ell'}}{n}\right), \exp\left(-2\mathrm{i}\pi\frac{s^{j'}_{\ell'+1}}{n}\right)\right]$$

do not cross. Further, as explained above, by construction, the chords of $\mathbf{P}(\mathcal{W}^{\circ})$ are chords between consecutive children of $\mathcal{T}^{\circ}(P)$. The equality $P = \mathbf{P}(\mathcal{W}^{\circ})$ then simply follows from the fact that, by construction and Proposition 2.1, $i, j \in [n]$ belong to the same block of P if and only if $u(i)$ and $u(j)$ have the same parent in $\mathcal{T}^{\circ}(P)$. See Figs. 8 and 9 for an example. □

As previously explained, we will prove the convergence, when $n \to \infty$, of a random non-crossing partition P_n of $[n]$ sampled according to $\mathbb{P}^{\mu}_{n}$, by looking at the scaling limit of the Łukasiewicz path of the conditioned Galton–Watson tree $\mathcal{T}^{\circ}(P_n)$. The latter is known (see Theorem 4.5 below) to be the normalized excursion of a spectrally positive strictly α-stable Lévy process X^{ex}_{α}, which we introduce next. The main advantage of this approach is that $\mathcal{T}^{\circ}(P_n)$ is a (conditioned) one-type Galton–Watson tree, whereas the dual tree $T^{\circ}(P_n)$ of P_n is a (conditioned) two-type Galton–Watson tree. We mention here that [1] uses a 'modified' Łukasiewicz path to study a two-type Galton–Watson tree; in fact this path is just the Łukasiewicz path of the one-type tree associated with the two-type tree by the Janson–Stefánsson bijection.

4.2. Convergence to the stable excursion

Fix $\alpha \in (1,2]$ and consider a strictly stable spectrally positive Lévy process of index α: X_α is a random process with paths in the set $\mathbb{D}([0,\infty),\mathbb{R})$ of *càdlàg* functions endowed with the Skorokhod J_1 topology (see *e.g.* Billingsley [9] for details on this space) which has independent and stationary increments, no negative jump and such that $\mathbb{E}[\exp(-\lambda X_\alpha(t))] = \exp(t\lambda^\alpha)$ for every $t, \lambda > 0$. Using excursion theory, it is then possible to define X_α^{ex}, the *normalized excursion of* X_α, which is a random variable with values in $\mathbb{D}([0,1],\mathbb{R})$, such that $X_\alpha^{\mathrm{ex}}(0) = X_\alpha^{\mathrm{ex}}(1) = 0$ and, almost surely, $X_\alpha^{\mathrm{ex}}(t) > 0$ for every $t \in (0,1)$. We do not enter into details, and refer the interested reader to Bertoin [8] for details on Lévy processes and Chaumont [11] for interesting ways to obtain such a process by path transformations.

An important point is that X_α^{ex} is continuous for $\alpha = 2$, and indeed $X_2^{\mathrm{ex}}/\sqrt{2}$ is the standard Brownian excursion, whereas the set of discontinuities of X_α^{ex} is dense in $[0,1]$ for every $\alpha \in (1,2)$; we shall treat the two cases separately. Duquesne [18, Proposition 4.3 and proof of Theorem 3.1] provides the following limit theorem, which is the stepping stone to our results in this section.

Theorem 4.5 (Duquesne [18]). *Fix $\alpha \in (1,2]$ and let $(\mu(k); k \geqslant 0)$ be a critical probability measure in the domain of attraction of a stable law of index α. For every integer n such that $\mathbb{Q}_{n+1}^\mu$ is well defined, sample τ_n according to $\mathbb{Q}_{n+1}^\mu$. Then there exists a sequence $(B_n)_{n\geqslant 1}$ of positive constants converging to ∞ such that the convergence*

$$\left(\frac{\mathcal{W}_{\lfloor ns\rfloor}(\tau_n)}{B_n}; 0 \leqslant s \leqslant 1\right) \xrightarrow[n\to\infty]{(d)} (X_\alpha^{\mathrm{ex}}(s); 0 \leqslant s \leqslant 1)$$

holds in distribution for the Skorokhod topology on $\mathbb{D}([0,1],\mathbb{R})$.

Recall that if we sample P_n according to $\mathbb{P}_n^\mu$, then the plane tree $\mathcal{T}^\circ(P_n)$ is distributed according to $\mathbb{Q}_{n+1}^\mu$. Thus, letting $\mathcal{W}^n = \mathcal{W}(\mathcal{T}^\circ(P_n))$ denote the Łukasiewicz path of $\mathcal{T}^\circ(P_n)$, the convergence

$$\left(\frac{\mathcal{W}^n_{\lfloor ns\rfloor}}{B_n}; 0 \leqslant s \leqslant 1\right) \xrightarrow[n\to\infty]{(d)} (X_\alpha^{\mathrm{ex}}(s); 0 \leqslant s \leqslant 1) \tag{4.2}$$

holds in distribution for the Skorokhod topology on $\mathbb{D}([0,1],\mathbb{R})$.

We next define continuous laminations by replacing the Łukasiewicz path by X_α^{ex} and mimicking the definition (4.1). We prove, using (4.2), that they are the limit of P_n as $n \to \infty$. We first consider the case $\alpha = 2$ as a warm-up before treating the more involved case $\alpha \in (1,2)$.

4.3. The Brownian case

Let $\mathbb{e} = X_2^{\mathrm{ex}}$; we define an equivalence relation $\overset{\mathbb{e}}{\sim}$ on $[0,1]$ as follows: for every $s, t \in [0,1]$, we set $s \overset{\mathbb{e}}{\sim} t$ when $\mathbb{e}(s \wedge t) = \mathbb{e}(s \vee t) = \min_{[s\wedge t, s\vee t]} \mathbb{e}$. We then define a subset of $\overline{\mathbb{D}}$ by

$$\mathbf{L}(\mathbb{e}) := \bigcup_{s \overset{\mathbb{e}}{\sim} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{4.3}$$

Using the fact that, almost surely, $\mathbb{e}$ is continuous and its local minima are distinct, one can prove (see Aldous [2] and Le Gall and Paulin [30]) that almost surely, $\mathbf{L}(\mathbb{e})$ is a geodesic lamination of $\overline{\mathbb{D}}$ and that, furthermore, it is maximal for the inclusion relation among geodesic laminations of $\overline{\mathbb{D}}$. Observe that $s \overset{\mathbb{e}}{\sim} s$ for every $s \in [0,1]$ so $\mathbb{S}^1 \subset \mathbf{L}(\mathbb{e})$. Also, since $\mathbf{L}(\mathbb{e})$ is maximal, its faces, *i.e.* the connected components of $\overline{\mathbb{D}} \setminus \mathbf{L}(\mathbb{e})$, are open triangles whose vertices belong to $\mathbb{S}^1$; $\mathbf{L}(\mathbb{e})$ is called the *Brownian triangulation* and corresponds to $\mathbf{L}_2$ in Theorem 4.1.

Proof of Theorem 4.1 for $\alpha = 2$. Using Skorokhod's representation theorem, we assume that the convergence (4.2) holds almost surely with $\alpha = 2$; we then fix ω in the probability space such that this convergence holds for ω. Since the space of compact subsets of $\overline{\mathbb{D}}$ equipped with the Hausdorff distance is compact, we have the convergence, along a subsequence (which depends on ω), of P_n to a limit L_∞, and it only remains to show that $L_\infty = \mathbf{L}(\mathbb{e})$. Observe first that, since the space of geodesic laminations of $\overline{\mathbb{D}}$ is closed, L_∞ is a lamination. Then, by maximality of $\mathbf{L}(\mathbb{e})$, it suffices to prove that $\mathbf{L}(\mathbb{e}) \subset L_\infty$ to obtain the equality of these two sets.

Fix $\varepsilon > 0$ and $0 \leqslant s < t \leqslant 1$ such that $s \overset{\mathbb{e}}{\sim} t$. Using the convergence (4.2) and the properties of the Brownian excursion (namely that times of local minima are almost surely dense in $[0,1]$), we can find integers $j_n, l_n \in \{1, \dots, n-1\}$ such that for every n large enough, we have $|n^{-1} j_n - s| < \varepsilon$ and $|n^{-1} l_n - t| < \varepsilon$ as well as

$$\mathcal{W}^n_{j_n} > \mathcal{W}^n_{j_n - 1} \quad \text{and} \quad l_n = \min\{m > j_n : \mathcal{W}^n_m < \mathcal{W}^n_{j_n}\}.$$

In other words, $u(j_n)$ and $u(l_n)$ are consecutive children of $u(j_n - 1)$ in $\mathcal{T}^\circ(P_n)$. By Proposition 4.4, the last display yields

$$\left[\exp\left(-2\mathrm{i}\pi \frac{j_n}{n}\right), \exp\left(-2\mathrm{i}\pi \frac{l_n}{n}\right)\right] \subset P_n.$$

Thus, for every n large enough, the chord $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ lies within distance 2ε from P_n. Letting $n \to \infty$, along a subsequence, we obtain that $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}]$ lies within distance 2ε from L_∞. As ε is arbitrary, we have $[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}] \subset L_\infty$, hence $\mathbf{L}(\mathbb{e}) \subset L_\infty$ and the proof is complete. □

4.4. The stable case

We follow the presentation of [27]. Fix $\alpha \in (1,2)$ and consider X^{ex}_α the normalized excursion of the α-stable Lévy process. For every $t \in (0,1]$, we let $\Delta X^{\mathrm{ex}}_\alpha(t) = X^{\mathrm{ex}}_\alpha(t) - X^{\mathrm{ex}}_\alpha(t-) \geqslant 0$ denote its jump at t, and we set $\Delta X^{\mathrm{ex}}_\alpha(0) = X^{\mathrm{ex}}_\alpha(0-) = 0$. We recall from [27, Proposition 2.10] that X^{ex}_α fulfils the following four properties with probability one.

(H1) For every $0 \leqslant s < t \leqslant 1$, there exists at most one value $r \in (s,t)$ such that $X^{\mathrm{ex}}_\alpha(r) = \inf_{[s,t]} X^{\mathrm{ex}}_\alpha$.

(H2) For every $t \in (0,1)$ such that $\Delta X^{\mathrm{ex}}_\alpha(t) > 0$, we have $\inf_{[t,t+\varepsilon]} X^{\mathrm{ex}}_\alpha < X^{\mathrm{ex}}_\alpha(t)$ for every $0 < \varepsilon \leqslant 1 - t$.

(H3) For every $t \in (0,1)$ such that $\Delta X^{\mathrm{ex}}_\alpha(t) > 0$, we have $\inf_{[t-\varepsilon,t]} X^{\mathrm{ex}}_\alpha < X^{\mathrm{ex}}_\alpha(t-)$ for every $0 < \varepsilon \leqslant t$.

(H4) For every $t \in (0,1)$ such that X^{ex}_α attains a local minimum at t (which implies $\Delta X^{\mathrm{ex}}_\alpha(t) = 0$ by (H3)), if $s = \sup\{u \in [0,t] : X^{\mathrm{ex}}_\alpha(u) < X^{\mathrm{ex}}_\alpha(t)\}$, then $\Delta X^{\mathrm{ex}}_\alpha(s) > 0$ and $X^{\mathrm{ex}}_\alpha(s-) < X^{\mathrm{ex}}_\alpha(t) < X^{\mathrm{ex}}_\alpha(s)$.

We will always implicitly discard the null set for which at least one of these properties does not hold. We next define a relation (not equivalence relation in general) on $[0,1]$ as follows: for every $0 \leqslant s < t \leqslant 1$, we set

$$s \simeq^{X_\alpha^{\mathrm{ex}}} t \quad \text{if } t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) \leqslant X_\alpha^{\mathrm{ex}}(s-)\},$$

and then for $0 \leqslant t < s \leqslant 1$ we set $s \simeq^{X_\alpha^{\mathrm{ex}}} t$ if $t \simeq^{X_\alpha^{\mathrm{ex}}} s$, and finally we agree that $s \simeq^{X_\alpha^{\mathrm{ex}}} s$ for every $s \in [0,1]$. We next define the following subset of $\overline{\mathbb{D}}$:

$$\mathbf{L}_\alpha := \bigcup_{s \simeq^{X_\alpha^{\mathrm{ex}}} t} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]. \tag{4.4}$$

Observe that $\mathbb{S}^1 \subset \mathbf{L}_\alpha$. Using the above properties, it is proved in [27, Proposition 2.9] that $\mathbf{L}_\alpha$ is a geodesic lamination of $\overline{\mathbb{D}}$, called the *α-stable lamination*. The latter is not maximal: each face is bounded by infinitely many chords (the intersection of the closure of each face and the unit disk has indeed a non-trivial Hausdorff dimension in the plane).

We next prove Theorem 4.1; as in the case $\alpha = 2$, we assume using Skorokhod's representation theorem that (4.2) holds almost surely, and we work with ω fixed in the probability space such that this convergence (as well as properties (H1)–(H4)) holds for ω. To simplify notation, we set

$$X^n(s) = \frac{1}{B_n} \mathcal{W}^n_{\lfloor ns \rfloor} \quad \text{for every } s \in [0,1].$$

Along a subsequence (which depends on ω), we have the convergence of P_n to a limit L_∞, which is a lamination. It only remains to prove the identity $L_\infty = \mathbf{L}_\alpha$. To do so, we shall prove the inclusions $\mathbf{L}_\alpha \subset L_\infty$ and $L_\infty \subset \mathbf{L}_\alpha$ in two separate lemmas.

Lemma 4.6. *We have $\mathbf{L}_\alpha \subset L_\infty$.*

Proof. Notice that if $s < t$ and $s \simeq^{X_\alpha^{\mathrm{ex}}} t$, then $X_\alpha^{\mathrm{ex}}(t) = X_\alpha^{\mathrm{ex}}(s-)$ and $X_\alpha^{\mathrm{ex}}(r) > X_\alpha^{\mathrm{ex}}(s-)$ for every $r \in (s,t)$, hence $s \simeq^{X_\alpha^{\mathrm{ex}}} t$ if and only if one of the following cases holds.

(i) If $\Delta X_\alpha^{\mathrm{ex}}(s) > 0$ and $t = \inf\{u > s : X_\alpha^{\mathrm{ex}}(u) = X_\alpha^{\mathrm{ex}}(s-)\}$, we write $(s,t) \in \mathcal{E}_1(X_\alpha^{\mathrm{ex}})$.

(ii) If $\Delta X_\alpha^{\mathrm{ex}}(s) = 0$, $X_\alpha^{\mathrm{ex}}(s) = X_\alpha^{\mathrm{ex}}(t)$ and $X_\alpha^{\mathrm{ex}}(r) > X_\alpha^{\mathrm{ex}}(s)$ for every $r \in (s,t)$, we write $(s,t) \in \mathcal{E}_2(X_\alpha^{\mathrm{ex}})$.

Using the observation ([27, Proposition 2.14]) that, almost surely, for every pair $(s,t) \in \mathcal{E}_2(X_\alpha^{\mathrm{ex}})$ and every $\varepsilon \in (0,(t-s)/2)$, there exists $s' \in [s, s+\varepsilon]$ and $t' \in [t-\varepsilon, t]$ with $(s',t') \in \mathcal{E}_1(X_\alpha^{\mathrm{ex}})$, one can prove ([27, Proposition 2.15]) that almost surely

$$\mathbf{L}_\alpha = \overline{\bigcup_{(s,t) \in \mathcal{E}_1(X_\alpha^{\mathrm{ex}})} \left[\mathrm{e}^{-2\mathrm{i}\pi s}, \mathrm{e}^{-2\mathrm{i}\pi t}\right]}. \tag{4.5}$$

The proof thus reduces to showing that, for any $0 \leqslant u < v \leqslant 1$ such that $\Delta X_\alpha^{\mathrm{ex}}(u) > 0$ and $v = \inf\{w \geqslant u : X_\alpha^{\mathrm{ex}}(w) = X_\alpha^{\mathrm{ex}}(u-)\}$ fixed, we have $[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}] \subset L_\infty$. Further, as in the case $\alpha = 2$, it is sufficient to find sequences $u_n \to u$ and $v_n \to v$ as $n \to \infty$ such that for every n large enough, $[\mathrm{e}^{-2\mathrm{i}\pi u_n}, \mathrm{e}^{-2\mathrm{i}\pi v_n}] \subset P_n$. Informally, the main difference from [27] is that we choose different sequences u_n, v_n: with the notation used in (4.1), we shall take the pair (u_n, v_n) of the form $n^{-1}(s_1^j, s_{k_j}^j)$ for a certain j.

More precisely, fix $\varepsilon > 0$ and observe that, since v cannot be a time for a local minimum of X_α^{ex} by (H4), then

$$\inf_{[v-\varepsilon,v+\varepsilon]} X_\alpha^{\mathrm{ex}} < X_\alpha^{\mathrm{ex}}(v) = X_\alpha^{\mathrm{ex}}(u-) < \inf_{[u,v-\varepsilon]} X_\alpha^{\mathrm{ex}}.$$

Using the convergence (4.2), we can then find a sequence $(u_n)_{n\geqslant 1}$ such that for every n sufficiently large, we have

$$u_n \in (u-\varepsilon, u+\varepsilon) \cap n^{-1}\mathbb{N} \quad \text{and} \quad \inf_{[v-\varepsilon,v+\varepsilon]} X^n < X^n(u_n-) < \inf_{[u_n,v-\varepsilon]} X^n.$$

Then define $v_n := \inf\{r \geqslant u_n : X^n(r) = X^n(u_n-)\}$ and observe that $v_n \in (v-\varepsilon, v+\varepsilon) \cap n^{-1}\mathbb{N}$. Moreover, as $B_n X^n(u_n) = \mathcal{W}^n_{nu_n}$ and $B_n X^n(u_n-) = \mathcal{W}^n_{nu_n-1}$, we have $\mathcal{W}^n_{nu_n-1} \leqslant \mathcal{W}^n_{nu_n}$ and

$$nv_n = \inf\{l \geqslant nu_n : \mathcal{W}^n_l = \mathcal{W}^n_{nu_n} - (\mathcal{W}^n_{nu_n} - \mathcal{W}^n_{nu_n-1})\}.$$

We conclude from Proposition 4.4 that

$$\left[\mathrm{e}^{-2\mathrm{i}\pi u_n}, \mathrm{e}^{-2\mathrm{i}\pi v_n}\right] \subset P_n$$

for every n large enough, and the proof is complete. □

Finally, we end the proof of Theorem 4.1 with the converse inclusion.

Lemma 4.7. *We have $L_\infty \subset \mathbf{L}_\alpha$.*

Proof. Recall that L_∞ is the limit of P_n along a subsequence, say, $(n_k)_{k\geqslant 1}$. Let us rewrite (4.1), combined with Proposition 4.4, as

$$P_{n_k} = \bigcup_{(u,v)\in\mathcal{E}_{(n_k)}} \left[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}\right],$$

where $\mathcal{E}_{(n_k)}$ is a symmetric finite subset of $[0,1]^2$. Upon extracting a further subsequence, we may, and do, assume that $\mathcal{E}_{(n_k)}$ converges in the Hausdorff sense as $k \to \infty$ to a symmetric closed subset $\mathcal{E}_\infty$ of $[0,1]^2$. One then checks that

$$L_\infty = \bigcup_{(u,v)\in\mathcal{E}_\infty} \left[\mathrm{e}^{-2\mathrm{i}\pi u}, \mathrm{e}^{-2\mathrm{i}\pi v}\right].$$

It only remains to prove that every pair $(u,v) \in \mathcal{E}_\infty$ satisfies $u \simeq^{X_\alpha^{\mathrm{ex}}} v$. Fix $(u,v) \in \mathcal{E}_\infty$ with $u < v$; we aim to show that $v = \inf\{r > u : X_\alpha^{\mathrm{ex}}(r) \leqslant X_\alpha^{\mathrm{ex}}(u-)\}$.

For every integer $j \in \{1,\ldots,n\}$, let $p(j)$ be the index of the parent of vertex labelled j in $\mathcal{T}^\circ(P_n)$: $p(j) = \sup\{m < j : \mathcal{W}^n_m \leqslant \mathcal{W}^n_j\}$. Observe then that $[\mathrm{e}^{-2\mathrm{i}\pi j_n/n}, \mathrm{e}^{-2\mathrm{i}\pi l_n/n}] \subset P_n$ when $p(j_n) = p(l_n)$ and, either $l_n = \inf\{m \geqslant j_n : \mathcal{W}^n_m = \mathcal{W}^n_{j_n} - 1\}$, or $j_n = p(j_n)+1$ and $l_n = \inf\{m \geqslant j_n : \mathcal{W}^n_m = \mathcal{W}^n_{p(j_n)}\}$.

By definition, (u,v) is the limit as $k \to \infty$ of elements (u_{n_k}, v_{n_k}) in $\mathcal{E}_{(n_k)}$. Upon extracting a subsequence, we may, and do, suppose that either each pair $(j_{n_k}, l_{n_k}) = (n_k u_{n_k}, n_k v_{n_k})$ fulfils the first condition above, or they all fulfil the second one. First we focus on the first case.

We therefore suppose that we can find integers $j_{n_k} < l_{n_k}$ in $\{1, \ldots, n_k\}$ such that

$$(u, v) = \lim_{k \to \infty} \left(\frac{j_{n_k}}{n_k}, \frac{l_{n_k}}{n_k} \right) \quad \text{and} \quad l_{n_k} = \inf\{m \geqslant j_{n_k} : \mathcal{W}_m^{n_k} = \mathcal{W}_{j_{n_k}}^{n_k} - 1\} \quad \text{for } k \geqslant 1.$$

We see that

$$X^{n_k}(r) \geqslant X^{n_k}\left(\frac{j_{n_k}}{n_k}\right) = X^{n_k}\left(\frac{l_{n_k} - 1}{n_k}\right) \quad \text{for every } r \in \left[\frac{j_{n_k}}{n_k}, \frac{l_{n_k} - 1}{n_k}\right], \tag{4.6}$$

which yields, together with the functional convergence $X^n \to X_\alpha^{\text{ex}}$,

$$X_\alpha^{\text{ex}}(r) \geqslant X_\alpha^{\text{ex}}(v-) \quad \text{for every } r \in (u, v). \tag{4.7}$$

By (H3), we must have $\Delta X_\alpha^{\text{ex}}(v) = 0$ and so $X^{n_k}(n_k^{-1}(l_{n_k} - 1)) \to X_\alpha^{\text{ex}}(v)$ as $k \to \infty$. On the other hand, the only possible accumulation points of $X^{n_k}(n_k^{-1} j_{n_k})$ are $X_\alpha^{\text{ex}}(u-)$ and $X_\alpha^{\text{ex}}(u)$.

We consider two cases. Suppose first that $\Delta X_\alpha^{\text{ex}}(u) = 0$; then $X^{n_k}(n_k^{-1} j_{n_k}) \to X_\alpha^{\text{ex}}(u)$ as $k \to \infty$ and it follows from (4.6) that $X_\alpha^{\text{ex}}(u) = X_\alpha^{\text{ex}}(v)$. This further implies that $X_\alpha^{\text{ex}}(u) < X_\alpha^{\text{ex}}(r)$ for every $r \in (u, v)$, otherwise it would contradict either (H1) or (H4), depending on whether X_α^{ex} admits a local minimum at u or not. We conclude that in this case, we have $u \simeq^{X_\alpha^{\text{ex}}} v$.

Suppose now that $\Delta X_\alpha^{\text{ex}}(u) > 0$; then, by (H2), for every $\varepsilon > 0$, there exists $r \in (u, u + \varepsilon)$ such that $X_\alpha^{\text{ex}}(r) < X_\alpha^{\text{ex}}(u)$. Consequently, we must have $X^{n_k}(n_k^{-1} j_{n_k}) \to X_\alpha^{\text{ex}}(u-)$ as $k \to \infty$, otherwise (4.6) would give $X_\alpha^{\text{ex}}(u) = X_\alpha^{\text{ex}}(v) = X_\alpha^{\text{ex}}(v-)$ and we would get a contradiction with (4.7). We thus have $X_\alpha^{\text{ex}}(u-) = X_\alpha^{\text{ex}}(v) \leqslant X_\alpha^{\text{ex}}(r)$ for every $r \in (u, v)$; moreover the latter inequality is strict since an element $r \in (u, v)$ such that $X_\alpha^{\text{ex}}(r) = X_\alpha^{\text{ex}}(u-)$ is the time of a local minimum of X_α^{ex}, and this contradicts (H4). We see again that $u \simeq^{X_\alpha^{\text{ex}}} v$.

In the second case when each pair (j_{n_k}, l_{n_k}) satisfies $j_{n_k} = p(j_{n_k}) + 1$ and

$$l_{n_k} = \inf\{m \geqslant j_{n_k} : \mathcal{W}_m^n = \mathcal{W}_{p(j_{n_k})}^n\},$$

the very same arguments apply, which completes the proof. □

5. Extensions

If P_n is a simply generated non-crossing partition generated using a sequence of weights w, a natural question is to ask how the largest block area of P_n behaves. In this direction, if P is a non-crossing partition, we propose studying $P^\bullet$, which is by definition the union of the convex hulls of the blocks of P (see Figure 10 for an example).

Question 1. *Assume that the weights w are equivalent to a critical probability distribution which has finite variance. Is it true that $P_n^\bullet$ converges in distribution as $n \to \infty$ to a random compact subset of the unit disk?*

If the answer was positive, the limiting object would be obtained from the Brownian triangulation by 'filling in' some triangles, and this would imply that the largest block area of P_n converges in distribution to the area of the largest 'filled-in face' of the distributional limit.

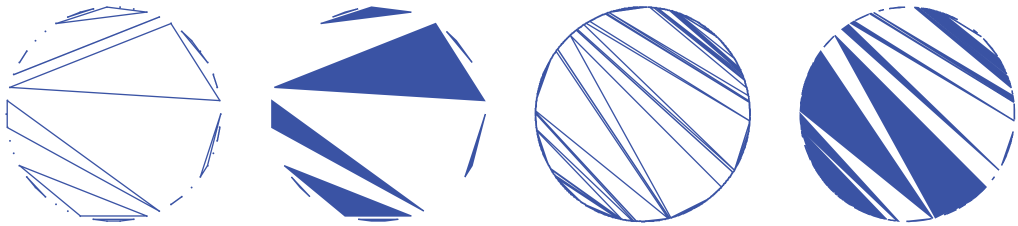


Figure 10. (a) P_{50}, (b) $P^{\bullet}_{50}$, (c) P_{500}, (d) $P^{\bullet}_{500}$, where P_{50} (resp. P_{500}) is a uniform non-crossing partition of [50] (resp. [500]).

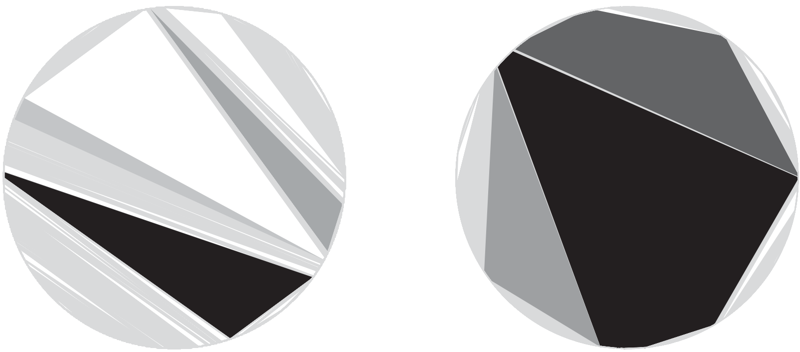


Figure 11. A simulation of $P^{\bullet}_{20000}$ for (a) $\alpha = 2$ and (b) $\alpha = 1.3$, where the largest faces are the darkest ones.

In the case of $\mathcal{A}$-constrained uniform plane partitions, numerical simulations based on the calculation of the total area of $P^{\bullet}_n$ indicate that this limiting distribution should depend on the weights (note that in the particular case $\mathcal{A} \subset \{1, 2\}$ it is clear that $(P_n, P^{\bullet}_n) \to (\mathbf{L}_2, \mathbf{L}_2)$ in distribution as $n \to \infty$).

When the weights w are equivalent to a critical probability distribution that belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2)$, it is not difficult to adapt the arguments of the previous section to check that

$$(P_n, \overline{\overline{\mathbb{D}} \setminus P^{\bullet}_n}) \quad \xrightarrow[n\to\infty]{(d)} \quad (\mathbf{L}_\alpha, \mathbf{L}_\alpha),$$

meaning that the faces of P_n cover in the limit the whole disk (see Figure 11 for an illustration). In particular, in this case, the largest block area of P_n converges in distribution to the largest area face of $\mathbf{L}_\alpha$.

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